Selected Solutions, Section 5.2

For problems 2, 5, 6, 8 do not spend too much time finding the general term(s) of the series. The recurrence relations are typically as far as we'll need to go.
In each of these problems, we take:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

2. In this case,

$$y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

Notice that the middle sum begins with x^1 rather than a constant (as the other sums do). We could simply begin that index with zero, write the first sum to match the other indices, then collect terms: Let $k = n-2$, or $n = k+2$ and:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k$$

Now,

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} - n a_n - a_n\right) x^n = \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} - (n+1)a_n\right) x^n$$

This gives us the recursion relation:

$$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0 \quad \Rightarrow \quad (n+2)(n+1)a_{n+2} = (n+1)a_n$$

$$a_{n+2} = \frac{a_n}{n+2}$$

Notice that

$$a_2 = \frac{1}{2}a_0 \quad a_3 = \frac{1}{3}a_1 \quad a_4 = \frac{1}{4}a_2 = \frac{1}{4\cdot 2}a_0 \quad a_5 = \frac{1}{5}a_3 = \frac{1}{5\cdot 3}a_1$$

$$a_6 = \frac{1}{6}a_4 = \frac{1}{6\cdot 4\cdot 2}a_0 \quad a_7 = \frac{1}{7}a_5 = \frac{1}{7\cdot 5\cdot 3}a_1 \quad a_8 = \frac{1}{8}a_6 = \frac{1}{8\cdot 6\cdot 4\cdot 2}a_0$$

and so on. We can write the solution $y(x)$ as:

$$y(x) = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{4\cdot 2}x^4 + \frac{1}{6\cdot 4\cdot 2}x^6 + \ldots\right) + a_1\left(x + \frac{1}{3}x^3 + \frac{1}{5\cdot 3}x^5 + \frac{1}{7\cdot 5\cdot 3}x^7 + \ldots\right)$$

These two functions (in series form) make up our fundamental set.

5. Follows much the same procedure:

$$(1-x)y'' + y = (1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

Multiply by the $1-x$ and incorporate the x into the sum:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

Now our difficulty is that the middle sum begins with x^1 but the others do not (beginning with $n=1$ would fix it). Also, the indices do not currently match. Start every sum with the same thing:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k$$

and

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1)k a_{k+1}x^k$$

Now we can write the differential equation as a single power series:

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + a_n\right] x^n$$

Giving us the recursion relation:

$$a_{n+2} = \frac{n}{n+2}a_{n+1} - \frac{1}{(n+2)(n+1)}a_n$$

In general, this means:

$$\begin{array}{lcll} a_2 & = & & -\frac{1}{2}a_0 \\ a_3 & = & \frac{1}{3}a_2 & -\frac{1}{6}a_1 \\ a_4 & = & \frac{1}{2}a_3 & -\frac{1}{12}a_2 \\ a_5 & = & \frac{3}{5}a_4 & -\frac{1}{20}a_3 \end{array}$$

and so on. To get our fundamental set, solve these first with $a_0 = 1, a_1 = 0$, then with $a_0 = 0$, $a_1 = 1$:

$$\begin{array}{ll} a_2 & = -\frac{1}{2} \\ a_3 & = -\frac{1}{6} \\ a_4 & = -\frac{1}{24} \end{array} \qquad \begin{array}{ll} a_2 & = 0 \\ a_3 & = -\frac{1}{6} \\ a_4 & = -\frac{1}{12} \\ a_5 & = -\frac{1}{24} \end{array}$$

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \dots\right) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \dots\right)$$

6. Goes much the same as Problem 5. Be sure to get your sums to all match in terms of beginning power of x and the index.

$$(2+x^2)y'' - xy' + 4y = (2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n$$

We'll try to manipulate the first sum to look like the second two:

$$(2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

The second sum begins with x^2, but beginning with zero might be OK, since the first two terms would be zero. Shift the index of the first sum to match ($k = n-2$ or $n = k+2$):

$$\sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} k(k-1)a_k x^k$$

Now we can collect all the terms together:

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} + n(n-1)a_n - na_n + 4a_n\right] x^k =$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} + (n^2-2n+4)a_n\right] x^k$$

This gives us the recursion:

$$a_{n+2} = -\frac{n^2-2n+4}{2(n+2)(n+1)}a_n$$

or:

$$a_2 = -a_0 \quad a_3 = -\frac{1}{4}a_1 \quad a_4 = -\frac{1}{6}a_2 = \frac{1}{6}a_0$$

$$a_5 = -\frac{7}{40}a_3 = \frac{7}{160}a_1 \qquad a_6 = -\frac{1}{5}a_4 = -\frac{1}{30}a_0$$

and so on. Writing y in terms of its fundamental set,

$$y(x) = a_0\left(1 - x^2 + \frac{1}{6}x^4 - \frac{1}{30}x^6 + \ldots\right) + a_1\left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \ldots\right)$$

8. Be careful in that our power series is now based at $x_0 = 1$ instead of $x_0 = 0$:

$$xy'' + y' + xy = x\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} + x\sum_{n=0}^{\infty} a_n(x-1)^n$$

The problem is that we cannot incorporate x into a series with an $(x-1)$ expansion. However, note that we can write

$$x = x - 1 + 1 \qquad \text{or } x = 1 + (x-1)$$

Making this substitution into the first sum,

$$(1+(x-1))\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1}$$

And similarly, into the last sum:

$$(1+(x-1))\sum_{n=0}^{\infty} a_n(x-1)^n = \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1}$$

Notice that our usual shift in the index won't work here- 2 of the sums start with $(x-1)^1$, the other 3 with constants. We will factor the constants out, and begin all indices at $n = 1$. Here are the five sums:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} = \sum_{n=1}^{\infty} (n+1)na_{n+1}(x-1)^n$$

$$\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1}(x-1)^n$$

$$\sum_{n=0}^{\infty} a_n(x-1)^n = a_0 + \sum_{n=1}^{\infty} a_n(x-1)^n$$

$$\sum_{n=0}^{\infty} a_n(x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1}(x-1)^n$$

Now simply collect it all into a single sum and extract the recursion:

$$(2a_2 + a_1 + a_0) +$$

$$\sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + (n+1)a_{n+1} + a_n + a_{n-1}\right](x-1)^n$$

with recursion:

$$a_{n+2} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+2)(n-1)} \qquad n = 1, 2, 3, \ldots$$

To get our fundamental set, we would first set $a_0 = 1$, $a_1 = 0$. We could then compute the coefficients to get $y_1(x)$. Next, set $a_0 = 0$, $a_1 = 1$ to get y_2 by computing the coefficients from our recursion.

15. I wanted you to work this through using our second technique for finding a series- That is, compute the derivatives directly (and then substitute them into the Taylor Series). In this case we want the first five non-zero terms- If the first 5 terms are non-zero, they would be:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4$$

Therefore, we only need to compute the derivatives from the DE: $y'' - xy' - y = 0$. Then we're given $y(0) = 2$, $y'(0) = 1$ and for the remaining terms:

$$y'' = xy' + y \quad \Rightarrow \quad y''(0) = 0 + y(0) = 2$$

$$y''' = y' + xy'' + y' = 2y' + xy'' \quad \Rightarrow \quad y'''(0) = 2y'(0) = 2$$

$$y^{(4)} = 2y'' + y'' + xy''' = 3y'' + xy''' \quad \Rightarrow \quad y^{(4)}(0) = 3y''(0) = 6$$

Therefore, the first five terms of the solution:

$$y(x) = 2 + x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 + \frac{6}{4!}x^4 + \cdots = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$$

Parts (b) and (c) need to be done on a computer (and are optional). For your convenience, I have plotted the 4 and 5 term expansions on the graph below using Maple.

If you've had Calc Lab, here is the Maple code I used:

```
Eqn15:=diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0;
inits:=y(0)=2, D(y)(0)=1;
IVP:={Eqn15,inits};
with(powseries):
f:=powsolve(IVP);
recursion_relation:=a(n)=subs(_k=n,f(_k));
F4:=tpsform(f,x,5);  #4th degree approx
F5:=tpsform(f,x,6); #5th degree approx
f4:=convert(F4,polynom,x);
f5:=convert(F5,polynom,x);
g:=rhs(dsolve(IVP,y(x)));
plot({g,f4,f5},x=-1..4,y=0..25);
```

And it gave me the plot in Figure 1.

16. Same as 15 (Directly compute the derivatives)

Given $(2 + x^2)y'' - xy' + 4y = 0$ with $y(0) = -1$ and $y'(0) = 3$ Then :

$$(2 + x^2)y'' - xy' + 4y = 0 \quad \Rightarrow \quad 2y''(0) - 0 + 4y(0) = 0 \quad \Rightarrow \quad y''(0) = -2y(0) = 2$$

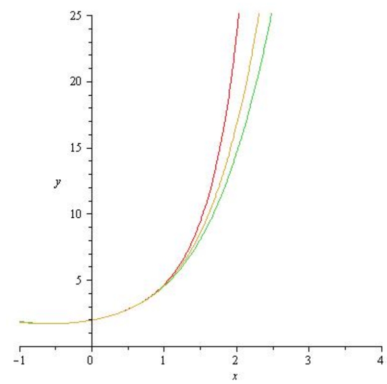

Figure 1: Order 4, 5, and the actual solution for Exercise 15.

For the third derivative, I would differentiate "in place" first, then simplify, then substitute $x = 0$:

$$(2+x^2)y'''+2xy''-xy''-y'+4y' = 0 \quad \Rightarrow (2+x^2)y'''+xy''+3y' = 0 \quad \Rightarrow \quad y'''(0) = -\frac{9}{2}$$

Similarly, after simplifying my next derivative:

$$(2 + x^2)y^{(4)} + 3xy^{(3)} + 4y'' = 0 \quad \Rightarrow \quad y^{(4)}(0) = -2y''(0) = -4$$

Therefore, the first 5 terms of the solution is:

$$y(x) = -1 + 3x + \frac{2}{2!}x^2 - \frac{9}{2 \cdot 3!}x^3 - \frac{4}{4!}x^4 + \cdots = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \cdots$$

(Again, parts (b) and (c) are optional since we need a computer) Using the same Maple commands (just changing the DE), we get the graph shown in Figure 2.

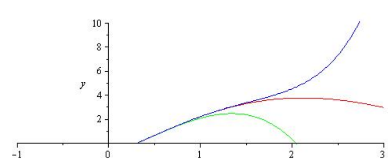

Figure 2: Order 4 (green) and 5 (blue) approximations and the solution (red) to the DE in Exercise 16. In this case, we'll need to increase the order of the solution by quite a bit in order to get better accuracy within this interval!

19. We'll go ahead and define $\dot{y}$ as dy/dt and $\ddot{y}$ and the second derivative of y with respect to t, where we make the subsitutions:

$$x + 1 = t \qquad \text{so that} \quad dx = dt \qquad \text{or} \qquad \frac{dx}{dt} = 1$$

We can also substitute:

$$(x-1)^2 = t^2 \qquad x^2 - 1 = (t+1)^2 - 1 = t^2 + 2t$$

Finally, we note that:

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \qquad \ddot{y} = y''(x)$$

Changing the DE to depend on t:

$$\ddot{y} + t^2\dot{y} + (t^2 + 2t)y = 0$$

Substitute the ansatz:

$$y = \sum_{n=0}^{\infty} a_n t^n \qquad \dot{y} = \sum_{n=1}^{\infty} n a_n t^{n-1} \qquad \dot{y} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

we get (I'm bringing in t^2 and $2t$ into the last sum, so note that there are 4 sums total):

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+2} + \sum_{n=0}^{\infty} 2 a_n t^{n+1} = 0$$

Checking these sums to make this a single polynomial, we see that the first power in each series: t^0, t^2, t^2, and t^1 respectively. We will make every sum begin with t^2. That means we need to break off the first two terms from the first sum:

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = 2a_2 + 6a_3 t + \sum_{n=4}^{\infty} n(n-1) a_n t^{n-2}$$

And one term from the last sum:

$$\sum_{n=0}^{\infty} 2 a_n t^{n+1} = 2a_0 t + \sum_{n=1}^{\infty} 2 a_n t^{n+1}$$

Now, all sums begin with t^2. We'll try to fill in the following sum:

$$2a_2 + (6a_3 + 2a_0)t + \sum_{k=2}^{\infty} \Big(\qquad\qquad\qquad\qquad \Big) t^k = 0$$

For the first sum, substitute $k = n - 2$ (or $n = k + 2$). For the second, third and fourth sums, we substitute $k = n - 1$ (or $n = k + 1$). Doing this, we get:

$$2a_2 + (6a_3 + 2a_0)t + \sum_{k=2}^{\infty} \left((k+2)(k+1)a_{k+2} + (k-1)a_{k-1} + a_{k-1} + 2a_{k+1}\right) t^k = 0$$

Which we can simplify a bit:

$$2a_2 + (6a_3 + 2a_0)t + \sum_{k=2}^{\infty} \left((k+2)(k+1)a_{k+2} + k a_{k-1} + 2a_{k+1}\right) t^k = 0$$

Set the coefficients to 0, and we get that a_0, a_1 are free variables, and

$$a_2 = 0, \qquad a_3 = -\frac{1}{3}a_0, \qquad a_{k+2} = -\frac{1}{k+2}a_{k-1} - \frac{1}{(k+2)(k+1)}a_{k-2}$$

where the recurrence works for $k = 2, 3, 4, \cdots$ (we already have a_3). Therefore, computing a_4, a_5 in terms of a_0, a_1, we get:

$$a_4 = -\frac{1}{4}a_1 - \frac{1}{4 \cdot 3}a_0$$

$$a_5 = -\frac{1}{5}a_2 - \frac{1}{5 \cdot 4}a_1 = 0 - \frac{1}{5 \cdot 4}a_1 = \frac{1}{20}a_1$$

$$a_6 = -\frac{1}{6}a_3 - \frac{1}{6 \cdot 5}a_2 = \frac{1}{6 \cdot 3}a_0$$

Therefore, factoring out a_0 and a_1, we get:

$$y = a_0 \left(1 - \frac{1}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{18}t^6 + \cdots\right) + a_1 \left(t - \frac{1}{4}t^4 + \frac{1}{20}t^5 + \cdots\right)$$

And we could get the desired result by back substituting $t = x - 1$ (to get a series solution in x).

20. Good practice with the Ratio Test. You should see that, for

$$y_1 = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \ldots (3n-1)(3n)}$$

the ratio of the $(n+1)^{\text{st}}$ term to the n^{th} term is:

$$\frac{|x|^3}{(3n+1)(3n+2)(3n+3)}$$

And for

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \ldots (3n)(3n+1)}$$

the ratio simplifies to:

$$\frac{|x|^3}{(3n+1)(3n+2)(3n+3)}$$

Thus, the limit of each is zero (so the radius of convergence is ∞).

23, 24 These are optional since they must be done in Maple.