Selected Solutions, Section 6.6

2. You can choose almost any function, even $1 * 1$:

$$1 * 1 = \int_0^t 1\, dx = \left. x \right|_0^t = t$$

3. The following trig identity is used[1]

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

Then:

$$\sin(t) * \sin(t) = \int_0^t \sin(t-x)\sin(x)\, dx = \frac{1}{2}\int_0^t \cos(t-2x) - \cos(t)\, dx$$

Use $u = t - 2x$, $du = -2\, dx$ for the first term:

$$\int \cos(t-2x)\, dx = -\frac{1}{2}\int \cos(u)\, du = -\frac{1}{2}\sin(t-2x)$$

The full antiderivative becomes:

$$\frac{1}{2}\left(-\frac{1}{2}\sin(t-2x) - x\cos(t)\right|_0^t = \frac{1}{2}\left[\left(-\frac{1}{2}\sin(-t) - t\cos(t)\right) - \left(-\frac{1}{2}\sin(t)\right)\right]$$

from which we get the textbook's answer: $(1/2)(\sin(t) - t\cos(t))$

4. We want to write:

$$\int_0^t (t-\tau)^2\cos(2\tau)\, d\tau \qquad \text{as} \qquad \int_0^t f(t-\tau)g(\tau)\, d\tau$$

In this case, we see that $f(t) = t^2$, $g(t) = \cos(2t)$, with corresponding Laplace transforms $F(s) = 2/s^3$ and $G(s) = s/(s^2+4)$. By the convolution theorem:

$$\mathcal{L}(f * g) = F(s)G(s) = \frac{2s}{s^3(s^2+4)} = \frac{2}{s^2(s^2+4)}$$

6. Same type as 4:

$$\int_0^t (t-\tau)\mathrm{e}^{\tau}\, d\tau = f * g$$

where $f(t) = t$, $g(t) = \mathrm{e}^t$, and corresponding Laplace transforms: $F(s) = 1/s^2$ and $G(s) = 1/(s-1)$. Therefore,

$$\mathcal{L}(f * g) = F(s)G(s) = \frac{1}{s^2(s-1)}$$

[1] I expected that you would probably need to look this up- Not necessary to memorize it.

8. The idea here is to write the given expression as $F(s)G(s)$, so that the inverse transform is $f * g$:

$$\frac{1}{s^4(s^2+1)} = F(s)G(s)$$

where

$$F(s) = \frac{1}{s^4} \qquad G(s) = \frac{1}{s^2+1}$$

Notice that to invert $F(s)$, we need to multiply and divide by $3! = 6$. Now,

$$\mathcal{L}^{-1}(F(s)G(s)) = \frac{1}{6}t^3 * \sin(t) = \frac{1}{6}\int_0^t (t-x)^3 \sin(x)\,dx$$

9. Same idea, you might group the s in the numerator with the s^2+4:

$$\frac{s}{(s+1)(s^2+4)} = F(s)G(s) \quad \text{where } F(s) = \frac{1}{s+1} \quad G(s) = \frac{s}{s^2+4}$$

Therefore,

$$\mathcal{L}^{-1}(F(s)G(s)) = f * g = \mathrm{e}^{-t} * \cos(2t) = \int_0^t \mathrm{e}^{-(t-x)} \cos(2x)\,dx$$

13.

$$y'' + \omega^2 y = g(t) \qquad y(0)=0 \quad y'(0)=1$$

$$s^2Y - 1 + \omega^2 Y = G(s) \quad \Rightarrow \quad (s^2+\omega^2)Y = G(s)+1 \quad \Rightarrow \quad Y = G(s)\frac{1}{s^2+\omega^2} + \frac{1}{s^2+\omega^2}$$

With:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{\omega}\frac{\omega}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin(\omega t)$$

Therefore, the solution is (the question asked us to write it as an integral):

$$y(t) = \frac{1}{\omega}\left(g(t) * \sin(\omega t) + \sin(\omega t)\right) = \frac{1}{\omega}\left(\int_0^t g(t-x)\sin(\omega x)\,dx + \sin(\omega t)\right)$$

14. Similar to Problem 13,

$$y'' + 2y' + 2y = \sin(\alpha t) \qquad \text{zero ICs}$$

$$(s^2+2s+2)Y = \frac{\alpha}{s^2+\alpha^2} \quad \Rightarrow$$

$$Y = \frac{\alpha}{s^2+\alpha^2}\cdot\frac{1}{s^2+2s+2} = \frac{\alpha}{s^2+\alpha^2}\cdot\frac{1}{(s+1)^2+1}$$

The inverse transform is the convolution of the inverses taken separately,

$$\mathcal{L}^{-1}\left(\frac{\alpha}{s^2+\alpha^2}\right) = \sin(\alpha t) \qquad \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) = \mathrm{e}^{-t}\sin(t)$$

so that:

$$y(t) = \sin(\alpha t) * \mathrm{e}^{-t}\sin(t) = \int_0^t \sin(\alpha(t-x))\mathrm{e}^{-x}\sin(x)\,dx$$

21. We did this problem in class as well- In this question, k is a function instead of the usual constant. Taking the Laplace transform of both sides, (then solve for $\Phi(s)$):

$$\Phi(s) + K(s)\Phi(s) = F(s) \quad \Rightarrow \quad \Phi(s) = \frac{F(s)}{1+K(s)}$$

22. (a) Same idea as 21:

$$\Phi(s) + \frac{\Phi(s)}{s^2} = \frac{2}{s^2+4} \quad \Rightarrow \quad \Phi(s)\left(\frac{s^2+1}{s^2}\right) = \frac{2}{s^2+4}$$

Therefore,

$$\Phi(s) = \frac{2s^2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

Doing the partial fractions, we get:

$$\begin{aligned} A + C &= 0 \\ 4A + C &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} B + D &= 2 \\ 4B + D &= 0 \end{aligned} \quad \Rightarrow \quad B = \frac{-2}{3}, D = \frac{8}{3}, A = 0, C = 0$$

Therefore,

$$\phi(t) = -\frac{2}{3}\sin(t) + \frac{4}{3}\sin(2t)$$

(b) **Some information that is needed:** How do you differentiate under the integral sign? Generally speaking,

$$\frac{d}{dt}\int_a^{h(t)} F(t,u)\,du = F(t,h(t))h'(t) + \int_a^{h(t)} F_t(t,u)\,du$$

The first part is the usual FTC, but the second part is probably not known to you (so it would be given as part of the problem). Given that,

$$\phi'(t) + (t-t)\phi(t) + \int_0^t \phi(u)\,du = 2\cos(2t)$$

Integrating a second time,

$$\phi''(t) + \phi(t) = -4\sin(2t)$$

For the initial conditions, go back to the original equation to see that $\phi(0) = 0$ and to our first derivative to see that $\phi'(0) = 2$

(c) To solve the IVP using Chapter 3 methods, the homogeneous part is

$$y_h(t) = C_1\sin(t) + C_2\cos(t)$$

And the particular part, use Method of Undetermined Coefficients:

$$y_p = A\cos(2t) + B\sin(2t) \quad \Rightarrow \quad y_p'' + y_p = -3A\cos(2t) - 3B\sin(2t) = -4\sin(2t)$$

so the full solution is the same as the one before.

23. (a) Solve by Laplace transform. You should get:

$$\Phi(s) = \frac{s}{s^2+1}$$

(b) The first derivative is:

$$\phi' + (t-t)\phi(t) + \int_0^t \phi(u)\,du = 0$$

so that the differential equation is

$$\phi''(t) + \phi(t) = 0$$

with ICs $\phi(0) = 1$ and $\phi'(0) = 0$ (from the equation for the derivative). This is simply the homogeneous solution-

26. (a) Solve by Laplace transform:

$$s\Phi - 0 + \Phi(s)\frac{1}{s^2} = \frac{1}{s^2} \qquad \Rightarrow \qquad \Phi(s) = \frac{1}{s^3+1}$$

Information you might find necessary: $s^3 + 1 = (s+1)(s^2 - s + 1)$
Therefore, using partial fractions, we find that

$$\Phi(s) = -\frac{1}{3}\frac{s-2}{s^2-s+1} + \frac{1}{3}\cdot\frac{1}{s+1}$$

(See if you can finish up from there).

(b) We need to differentiate twice to get rid of the integral: $\phi''' + \phi = 0$.
For ICs, we see that $\phi(0) = 0$ was given, then from the original equation, $\phi'(0) = 0$ and from our first derivative, $\phi''(0) = 1$.

(c) To continue, from the previous section, we know that the characteristic equation factors as:

$$(r+1)(r^2 - r + 1) = 0$$

From which we get one real value, $r = -1$, and two complex values, $r = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.
From Chapter 3, I think we would probably guess that the solution is therefore

$$\phi(t) = C_1 e^{-t} + e^{t/2}\left(C_2 \cos(\sqrt{3}/2\,t) + C_3 \sin(\sqrt{3}/2\,t)\right)$$

which is hopefully what we got earlier.