Selected Solutions: Section 6.1

1. This is piecewise continuous, but not continuous at $t = 1$.

2. Not continuous and not piecewise continuous.

3. Continuous (so also piecewise continuous).

5. (a) Find the Laplace transform of t (done in class).

   (b) Find the Laplace transform of t^2:

$$\mathcal{L}(t^2) = \int_0^\infty \mathrm{e}^{-st} t^2 \, dt$$

which is integrated by parts:

$$\begin{array}{crr} + & t^2 & \mathrm{e}^{-st} \\ - & 2t & -(1/s)\mathrm{e}^{-st} \\ + & 2 & (1/s^2)\mathrm{e}^{-st} \\ - & 0 & -(1/s^3)\mathrm{e}^{-st} \end{array} \quad \Rightarrow \quad \lim_{T\to\infty} -\mathrm{e}^{-st}\left(\frac{s^2t^2+2st+2}{s^3}\right)\Bigg|_0^T = 0 + \frac{2}{s^3}, \quad s > 0$$

NOTE: The limit is zero because

$$\lim_{t\to\infty} t^n \mathrm{e}^{-st} = 0$$

for any $n = 0, 1, 2, 3, \cdots$ and $s > 0$ (by l'Hospital's rule). You should include a note like this for your justification (unless you compute out the limit).

21. Recall that the inverse tangent function has a limit as $t \to \infty$; the function approaches $\pi/2$ (which is a vertical asymptote for the original tangent).

23. This one is a little tricky in that you do NOT want to compute the antiderivative (the antiderivative is not an "elementary" function- meaning we would need a series representation. Rather, we note that

$$t^{-2}\mathrm{e}^t = \frac{\mathrm{e}^t}{t^2}$$

which diverges (to infinity). Therefore, the integral given will diverge as well.

26. The Gamma Function $\Gamma(p)$ is an extension of the factorial function to non-integers. In this exercise, we show that the Gamma function, when restricted to the integers, gives the factorial.

(a) If $p > 0$, then show $\Gamma(p+1) = p\Gamma(p)$:

$$\Gamma(p+1) = \int_0^\infty \mathrm{e}^{-x} x^p \, dx$$

Integration by parts gives us the answer for $p > 0$. Actually, the following is true for $p > -1$:

$$\begin{array}{c|c|c} + & x^p & \mathrm{e}^{-x} \\ - & px^{p-1} & -\mathrm{e}^{-x} \end{array} \Rightarrow -x^p \mathrm{e}^{-x}\Big|_0^\infty + p\int_0^\infty \mathrm{e}^{-x} x^{p-1} \, dx$$

The quantity $-x^p\mathrm{e}^{-x}$ goes to zero as $x \to \infty$ for any p. However, if p is negative we have to be careful about x^p as $x \to 0$. If we restrict $p > 0$, then $x^p\mathrm{e}^{-x} = 0$ at zero, and we get:

$$\Gamma(p+1) = \int_0^\infty \mathrm{e}^{-x} x^p \, dx = p\int_0^\infty \mathrm{e}^{-x} x^{p-1} \, dx = p\Gamma(p)$$

(b) Show that $\Gamma(1) = 1$. We can do this directly by taking $p = 0$:

$$\int_0^\infty \mathrm{e}^{-x} \, dx = -\mathrm{e}^{-x}\Big|_0^\infty = 0 - -1 = 1$$

(c) If p is a positive integer, show that $\Gamma(n+1) = n!$.
We can show this by induction. We note from parts (a) and (b) that:

$$\Gamma(1) = 1 \quad \Gamma(2) = 1 \cdot \Gamma(1) = 1 \quad \Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1$$

In this case, we showed that the formula works if $n = 1, 2$ or 3 (not necessary, but it does give you a general idea).
Assume that the formula works for $n = k$, $\Gamma(k+1) = k!$. Show that it works for $n = k+1$. By Part (a),

$$\Gamma(k+2) = (k+1)\Gamma(k+1)$$

And by what we assumed, if $k+2$ is a positive integer, then

$$\Gamma(k+2) = (k+1)\Gamma(k+1) = (k+1)k! = (k+1)!$$

Therefore, we have proved by induction that $\Gamma(n+1) = n!$

(d) (This part can be omitted) By repeating the process in (c),

$$\Gamma(p+n) = p\Gamma(p+n-1) = (p+n-1)(p+n-2)\Gamma(p+n-2) =$$

$$= \ldots = p(p+1)(p+2)\cdots(p+n-1)\Gamma(p)$$

27. We typically won't use the Gamma function, but this exercise helps us to understand Table Entry #4 a little better (in the Table of Laplace transforms).

(a) Hint: Let $x = st$, then do a change of variables.

(b) Straightforward- Use the result of 26.

(c) This is an interesting problem, but may be omitted. Assuming the formulas given in the text,

$$\mathcal{L}(t^{-1/2}) = \int_0^\infty \mathrm{e}^{-st} \frac{1}{\sqrt{t}}\, dt$$

Looking at what we want, we'll try setting $x^2 = st$ and perform a substitution. Finding dx and dt, we get:

$$2x\, dx = s\, dt \quad \Rightarrow \quad 2\sqrt{st}\, dx = s\, dt \quad \Rightarrow \quad \frac{2}{\sqrt{s}}\, dx = \frac{1}{\sqrt{t}}\, dt$$

which is what we needed to get the expression in the text:

$$\mathcal{L}(t^{-1/2}) = \frac{2}{\sqrt{s}} \int_0^\infty \mathrm{e}^{-x^2}\, dx = \sqrt{\frac{\pi}{s}}$$

(d) Finally, we'll use the result from 26: $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2)$ to compute this:

$$\mathcal{L}(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$