Section 6.3 Homework Notes

2. To sketch the graph, try first rewriting what is given as a piecewise defined function. The function:

$$(t-3)u_2(t) - (t-2)u_3(t)$$

depends on the value of t:

- If $t < 2$, the function is zero, since u_2 and u_3 are zero.
- If $2 \leq t < 3$, then the function is just $t-3$, since $u_3(t)$ is still zero.
- If $t \geq 3$, then the function would be $t-3-(t-2) = -3+2 = -1$, since both u_2 and u_3 would simplify to 1.

Now it's easy to graph.

3. Shift the right half of the graph of t^2 to the right π units.

6. Break it up:

$$(t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t) = \begin{cases} 0 & \text{if } t < 1 \\ (t-1) & \text{if } 1 \leq t < 2 \\ (t-1) - 2(t-2) & \text{if } 2 \leq t < 3 \\ (t-1) - 2(t-2) + (t-3) & \text{if } t \geq 3 \end{cases}$$

Rewriting this, we get:

$$= \begin{cases} 0 & \text{if } t < 0 \\ t-1 & \text{if } 1 \leq t < 2 \\ -t+3 & \text{if } 2 \leq t < 3 \\ 0 & \text{if } t \geq 3 \end{cases}$$

8. We could do this one piecemeal:

- Start things off at 1.
- Next, we need to go to -1 (2 units down), so take $1 - 2u_1(t)$
- Next, get back to 1 (2 units up) at $t = 2$: $1 - 2u_1(t) + 2u_2(t)$
- At time 3, back to -1 (2 units down): $1 - 2u_1(t) + 2u_2(t) - 2u_3(t)$
- At time 4, back to zero (1 unit up): $1 - 2u_1(t) + 2u_2(t) + 2u_3(t) + u_4(t)$

Alternative: Use the on-off switch and simplify:

$$(1 - u_1(t)) - (u_1 - u_2(t)) + (u_2(t) - u_3(t)) - (u_3(t) - u_4(t))$$

11. In this case, best to use the on-off switch and simplify:

$$t(1-u_1(t)) + (t-1)(u_1(t) - u_2(t)) + (t-2)(u_2(t) - u_3(t)) = t - u_1(t) - u_2(t) - (t-2)u_3(t)$$

14. To use the table, we need that:

$$u_1(t)f(t-1) = u_1(t)(t^2 - 2t + 2)$$

so that $f(t-1) = t^2 - 2t + 2$. This means that

$$f(t) = (t+1)^2 - 2(t+1) + 2 = t^2 + 2t + 1 - 2t - 2 + 2 = t^2 + 1$$

Now the table entry says:

$f(t)$	$F(s)$
$u_c(t)f(t-c)$	$\mathrm{e}^{-sc}F(s)$

so we see that

$$F(s) = \mathcal{L}(t^2 + 1) = \frac{2}{s^3} + \frac{1}{s}$$

so that our final answer is: $\mathrm{e}^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$

15. Same idea as 14:

$$f(t) = (t - \pi)(u_\pi(t) - u_{2\pi}(t)) = (t-\pi)u_\pi(t) - (t-\pi)u_{2\pi}(t)$$

Inverting term by term, the first is:

$$u_\pi(t)(t-\pi) = u_\pi(t)g(t-\pi) \quad \Rightarrow \quad g(t) = t \quad \Rightarrow \quad \frac{\mathrm{e}^{-\pi s}}{s^2}$$

and for the second term, think of

$$u_{2\pi}(t)(t-\pi) = u_\pi(t)g(t-2\pi) \quad \Rightarrow \quad g(t) = t + \pi \quad \Rightarrow \frac{\mathrm{e}^{-2\pi s}}{s^2} + \frac{\pi \mathrm{e}^{-2\pi s}}{s}$$

19. Use Table Entry 11.

20. Find the inverse Laplace transform of

$$F(s) = \mathrm{e}^{-2s} \cdot \frac{1}{s^2 + s - 2}$$

This is of the form $\mathrm{e}^{-sc}F(s)$, so we need to find the inverse transform of $1/(s^2 + s - 2)$. We'll do this by Partial Fraction Decomposition. This F refers to the table, not the original F:

$$F(s) = \frac{1}{s^2 + s - 2} = \frac{A}{s+2} + \frac{B}{s-1} = -\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}$$

and

$$f(t) = -\frac{1}{3}\mathrm{e}^{-2t} + \frac{1}{3}\mathrm{e}^{t}$$

So our overall answer is $u_2(t)f(t-2)$, or:

$$u_2(t)\left(-\frac{1}{3}\mathrm{e}^{-2(t-2)} + \frac{1}{3}\mathrm{e}^{t-2}\right)$$

21. Find the inverse Laplace transform of

$$G(s) = \frac{2e^{-2s}(s-1)}{s^2 - 2s + 2}$$

(I changed the notation of the original function so as not to confuse $F(s)$ in the table with $F(s)$ in the original question).

We will rewrite this expression, keeping the table in mind:

$$G(s) = 2e^{-2s}\frac{s-1}{s^2-2s+2} = 2e^{-2s}\frac{s-1}{(s-1)^2+1} = 2e^{-sc}F(s)$$

We see that, given this $F(s)$, then $f(t) = e^t \cos(t)$ and our overall inverse Laplace transform is:

$$2u_2(t)f(t-2) = 2u_2(t)e^{t-2}\cos(t-2)$$

25. Good practice to work from the definition, and also to look at where we want to go: We want to show that

$$\mathcal{L}(f(ct)) = \int_0^\infty e^{-st} f(ct)\, dt = \frac{1}{c}\int_0^\infty e^{-s/c} f(w)\, dw$$

So, it looks like we want to do a substitution: $w = ct$, so $dw = c\,dt$ and $dt = \frac{1}{c}dw$.

For part (b), use part (a)-

$$f(ct) = \mathcal{L}^{-1}\left(\frac{1}{c}F(s/c)\right) \quad \Rightarrow \quad cf(ct) = \mathcal{L}^{-1}\left(F(s/c)\right)$$

Then let $c = \frac{1}{k}$.

For part (c), use these ideas:

- From part (b), $\mathcal{L}^{-1}(F(as)) = \frac{1}{a}f(t/a)$
- Note that $as + b = a(s + b/a)$
- Use the table to see that $\mathcal{L}(e^{ct}f(t)) = F(s-c)$

27. We want to try to use the result of Exercise 25 to solve this one, so put it in the form $F(s) = G(as+b)$.

$$F(s) = \frac{2s+1}{4s^2+4s+5} = \frac{2s+1}{(4s^2+4s+1)+4} = \frac{2s+1}{(2s+1)^2+4} = G(2s+1)$$

where $G(s) = \frac{s}{s^2+4}$ and so $g(t) = \cos(2t)$. Using the results of 25, with $a = 2$ and $b = 1$,

$$\mathcal{L}^{-1}(G(2s+1)) = \frac{1}{2}e^{-t/2}\cos(2t/2) = \frac{1}{2}e^{-t/2}\cos(t)$$

28. Similar to 27- Put this in the form $G(as+b)$. We might note that $(3s-2)^2 = 9s^2-12s+4$. Therefore,

$$\frac{1}{9s^2-12s+3} = \frac{1}{(3s-2)^2-1} = G(3s-2)$$

so that $G(s) = 1/(s^2-1)$ and $g(t) = \sinh(t)$ (You can use the hyperbolic functions from the table).

Therefore, with $a = 3$ and $b = -2$,

$$\mathcal{L}^{-1}(G(3s-2)) = \frac{1}{3}\mathrm{e}^{2t/3}\sinh(t/3)$$

32. For this problem, we'll need to have a sum for a finite geometric series. Here's one way to get that formula:

$$\begin{array}{rl} S & = 1 + r + r^2 + \cdots + r^m \\ rS & = r + r^2 + \cdots + r^{m+1} \\ \hline (1-r)S = 1 - r^{m+1} & \end{array} \quad \Rightarrow \quad S = \frac{1-r^{m+1}}{1-r}$$

Now, back to the text. **Note that**, if $t > 0$, then $u_0(t) = 1$.

The Laplace transform is a linear operator, so:

$$\mathcal{L}\left(\sum_{k=0}^{2n+1}(-1)^k u_k(t)\right) = \sum_{k=0}^{2n+1}(-1)^k\mathcal{L}(u_k(t)) = \sum_{k=0}^{2n+1}(-1)^k\frac{\mathrm{e}^{-ks}}{s} = \frac{1}{s}\sum_{k=0}^{2n+1}\left(-\mathrm{e}^{-s}\right)^k = \frac{1}{s}\cdot\frac{1-(-\mathrm{e}^{-s})^{2n+2}}{1+\mathrm{e}^{-s}}$$

This reminds us of a geometric series

33. Pretty much the same idea as 32, except we take the infinite sum:

$$\mathcal{L}\left(\sum_{k=0}^{\infty}(-1)^k u_k(t)\right) = \sum_{k=0}^{\infty}(-1)^k\mathcal{L}(u_k(t)) = \sum_{k=0}^{\infty}(-1)^k\frac{\mathrm{e}^{-ks}}{s}$$

This reminds us of a geometric series with $r = -\mathrm{e}^{-s}$ (except now it is an infinite sum.

35. Use the results of Problem 34- If f is periodic with period T, then

$$\mathcal{L}(f(t)) = \frac{\int_0^T \mathrm{e}^{-st}f(t)\,dt}{1-\mathrm{e}^{-sT}}$$

In this case, $T = 2$, and

$$\int_0^2 \mathrm{e}^{-st}f(t)\,dt = \int_0^1 \mathrm{e}^{-st}\,dt = -\frac{1}{s}\mathrm{e}^{-st}\Big|_0^1 = -\frac{\mathrm{e}^{-s}}{s} + \frac{1}{s} = \frac{1}{s}\left(1-\mathrm{e}^{-s}\right)$$

Put this into the formula:

$$\mathcal{L}(f(t)) = \frac{\frac{1}{s}(1-\mathrm{e}^{-s})}{1-\mathrm{e}^{-2s}}$$

This may not seem to be the same answer as in Problem 33; However note that:

$$1-\mathrm{e}^{-2s} = 1^2 - \left(\mathrm{e}^{-s}\right)^2 = \left(1-\mathrm{e}^{-s}\right)\left(1+\mathrm{e}^{-s}\right)$$