Section 6.4 Homework Notes

2. We'll solve by hand and use Wolfram Alpha to plot the solution (command given below):

$$y'' + 2y' + 2y = u_\pi(t) - u_{2\pi}(t), \quad y(0) = 0, \quad y'(0) = 1$$

Take the Laplace transforms and solve for $Y(s)$:

$$(s^2Y - 0 - 1) + 2(sY - 0) + 2Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s}$$

$$(s^2 + 2s + 2)Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s} + 1$$

$$Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s(s^2+2s+2)} + \frac{1}{s^2+2s+2}$$

We'll do the last term first:

$$\frac{1}{s^2+2s+2} = \frac{1}{s^2+2s+1+1} = \frac{1}{(s+1)^2+1}$$

so the inverse Laplace transform is (table entry 19): $\mathrm{e}^{-t}\sin(t)$.

Next, notice that the first term is of the form:

$$\left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s(s^2+2s+2)} = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)H(s)$$

So if we find $h(t)$, the inverse Laplace transform of this part will be:

$$u_\pi(t)h(t-\pi) - u_{2\pi}(t)h(t-2\pi)$$

Therefore, we only need to focus on inverting:

$$H(s) = \frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

Doing the partial fractions, we get:

$$1 = A(s^2+2s+2) + (Bs+C)s$$

If $s = 0$, then $A = 1/2$. Substituting this in, and simplifying the result, we get:

$$1 = \frac{1}{2}s^2 + s + 1 + (Bs^2 + Cs) \quad \Rightarrow \quad -\frac{1}{2}s^2 - s = Bs^2 + Cs$$

so $B = -1/2$ and $C = -1$. Put it all together:

$$\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s+2}{s^2+2s+2} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\left[\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right]$$

Now $h(t) = \frac{1}{2} - \frac{1}{2}\left(\mathrm{e}^{-t}\cos(t) + \mathrm{e}^{-t}\sin(t)\right)$. Putting it all together,

$$y(t) = \mathrm{e}^{-t}\sin(t) + u_\pi(t)h(t-\pi) - u_{2\pi}(t)h(t-2\pi)$$

The plot in Wolfram Alpha is a little hard to read. Here is a clearer graph of the solution, with the Heaviside function overlaying it:

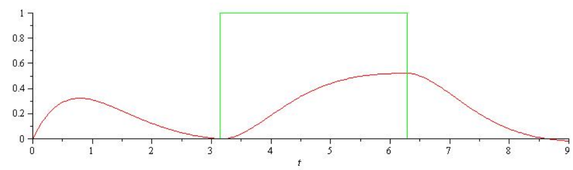

Figure 1: Solution and Heaviside function for Exercise 2.

4. Take the Laplace transform of both sides to get:

$$(s^2+4)Y = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} \quad \Rightarrow \quad Y = \frac{1+e^{-\pi s}}{(s^2+1)(s^2+4)}$$

Using Partial Fractions:

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

Multiply by the denominator and simplify to get:

$$1 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

Therefore, we have 4 equations in 4 unknowns:

$$A+C=0 \qquad B+D=0 \qquad 4A+C=0 \qquad 4B+D=1$$

Using the first and third equations, $A=C=0$.

Using the second and fourth equations, $B=1/3$, and $D=-1/3$.

You can think of 1 as e^{-0s}, so that Y is of the form:

$$Y(s) = (e^{-as} - e^{-bs})H(s)$$

whose inverse Laplace transform is:

$$y(t) = u_a(t)h(t-a) - u_b(t)h(t-b)$$

Therefore, in this case, $a=0, b=\pi$, and

$$h(t) = \frac{1}{3}\sin(t) + \frac{1}{3}\sin(2t)$$

so that

$$y(t) = u_0(t)h(t-0) + u_{2\pi}(t)h(t-2\pi) = h(t) + u_{2\pi}(t)h(t-2\pi)$$

The graphical part of this is optional, but informative. If we plot both the forcing function $\sin(t) - u_\pi(t)\sin(t-\pi)$ with the solution to the DE using the Maple software:

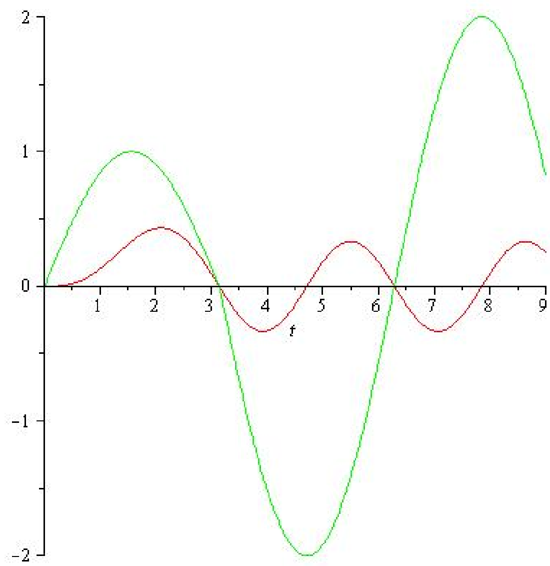

Figure 2: Forcing function (green) and solution (red) to the DE in Exercise 4.

```
with(plots):
Eqn04:=diff(y(t),t$2)+4*y(t)=sin(t)+Heaviside(t-Pi)*sin(t-Pi);
Y1:=dsolve({Eqn04,y(0)=0,D(y)(0)=0},y(t));
A:=plot(rhs(Y1),t=0..9,color=red);
B:=plot(sin(t)-Heaviside(t-Pi)*sin(t-Pi),t=0..9, color=green);
display(A,B);
```

We get graph in Figure 2.

6.

$$y'' + 3y' + 2y = u_2(t), \qquad y(0) = 0, \quad y'(0) = 1$$

As is usual, take the Laplace transforms and solve for $Y(s)$:

$$(s^2Y - 0 - 1) + 3(sY - 0) + 2Y = \mathrm{e}^{-2s}\frac{1}{s}$$

$$Y = \mathrm{e}^{-2s}\frac{1}{s(s^2 + 3s + 2)} + \frac{1}{s^2 + 3s + 2}$$

Break it up- Let's do the second term first:

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{(s + 2)(s + 1)} = \frac{A}{s + 2} + \frac{B}{s + 1} = -\frac{1}{s + 2} + \frac{1}{s + 1}$$

so the inverse Laplace transform of this part is: $-\mathrm{e}^{-2t} + \mathrm{e}^{-t}$.

Alternative Solution to this part. We could have completed the square in the denominator:

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{s^2 + 3s + \frac{9}{4} + 2 - \frac{9}{4}} = 2\frac{\frac{1}{2}}{\left(s + \frac{3}{2}\right)^2 - \frac{1}{4}}$$

Combine table entries 14 and 7 to get the inverse Laplace transform as:

$$2\mathrm{e}^{-\frac{3}{2}t}\sinh\left(\frac{1}{2}t\right)$$

This is the solution that Maple gives you. Notice that it is the same as our solution:

$$2\mathrm{e}^{-\frac{3}{2}t}\sinh\left(\frac{1}{2}t\right) = 2\mathrm{e}^{-\frac{3}{2}t}\cdot\frac{1}{2}\left(\mathrm{e}^{\frac{1}{2}t}-\mathrm{e}^{-\frac{1}{2}t}\right) = \mathrm{e}^{-t}-\mathrm{e}^{-2t}$$

Optional: We can do the graphical analysis using the Maple software

```
with(plots):
Eqn06:=diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=Heaviside(t-2);
Y1:=dsolve({Eqn06,y(0)=0,D(y)(0)=1},y(t));
A:=plot(rhs(Y1),t=0..9,color=red);
B:=plot(Heaviside(t-2),t=0..9, color=green);
display(A,B);
```

which gives us the graph in Figure 3.

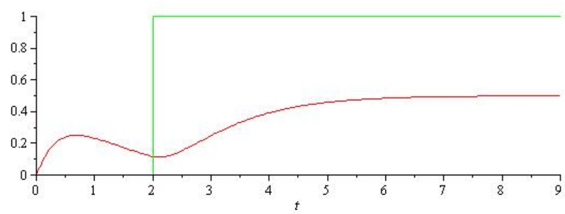

Figure 3: Forcing function (green) and solution (red) to the DE in Exercise 6.

8. Same idea as before. Once we take the Laplace transforms of both sides, we get:

$$\left(s^2+s+\frac{5}{4}\right)Y = \frac{1}{s^2}-\frac{\mathrm{e}^{-\pi/s}}{s^2} = \frac{1-\mathrm{e}^{-\pi/s}}{s^2}$$

so that

$$Y = \frac{1-\mathrm{e}^{-\pi/s}}{s^2(s^2+s+5/4)}$$

Think of this as:

$$Y = (\mathrm{e}^{-as}-\mathrm{e}^{-bs})H(s)$$

(with $a=0$) so that, if we find $h(t)$, our overall solution will be:

$$y(t) = u_0(t)h(t-0) - u_b(t)h(t-b) = h(t) - u_b(t) - h(t-b)$$

In this case, $a=0, b=\pi$ and

$$H(s) = \frac{1}{s^2(s^2+s+5/4)} = \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+s+5/4}$$

Solving for the constants, we get

$$As(s^2 + s + 5/4) + B(s^2 + s + 5/4) + s^2(Cs + D) = 1$$

Taking $s = 0$, we immediately get $B = 4/5$, which we then substitute in and simplify to get:

$$(A + C)s^3 + (A + D)s^2 + \frac{5A}{4}s = -\frac{4}{5}s^2 - \frac{4}{5}s$$

from which:

$$A = -\frac{16}{25} \quad B = \frac{4}{5} \quad C = \frac{16}{25} \quad D = \frac{-4}{25}$$

Therefore, we need to work with the last term, and factor out what we can:

$$\frac{\frac{16}{25}s - \frac{4}{25}}{s^2 + s + 5/4} = \frac{4}{25}\left[\frac{4(s + 1/2)}{(s + 1/2)^2 + 1} - 3\frac{1}{(s + 1/2)^2 + 1}\right]$$

Now,

$$h(t) = -\frac{16}{25} + \frac{4}{5}t + \frac{4}{25}e^{-t/2}\left[4\cos(t) - 3\sin(t)\right]$$

And the solution to the DE is:

$$y(t) = h(t) - u_{\pi/2}(t)h(t - \pi/2)$$

Optionally, here is the Maple software code to get the answer and plot the solution, which we do below.

```
with(plots):
Eqn08:=diff(y(t),t$2)+diff(y(t),t)+(5/4)*y(t)=t-Heaviside(t-Pi/2)*(t-Pi/2);
Y1:=dsolve({Eqn08,y(0)=0,D(y)(0)=0},y(t));
A:=plot(rhs(Y1),t=0..9,color=red);
B:=plot(t-(t-Pi/2)*Heaviside(t-Pi/2),t=0..9, color=green);
display(A,B);
```

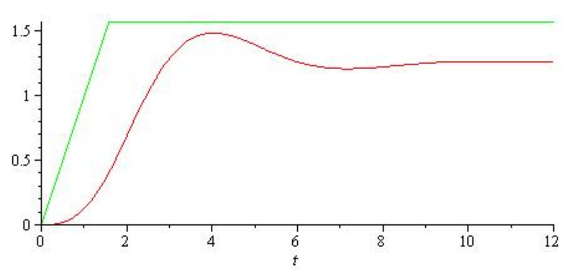

Figure 4: Forcing function (green) and solution (red) to the DE in Exercise 8.

14. We want a function g that ramps up from the point $(t_0, 0)$ to $(t_0 + k, h)$ and then stays at that value. The line segment has the equation: $y = \frac{h}{k}(t - t_0)$, so overall,

$$f(t) = \frac{h}{k}(t - t_0)(u_{t_0}(t) - u_{t_0+k}(t)) + \frac{h}{k}u_{t_0+k}$$

15. Starts the same as 14, but ramps back down to zero:

- Ramps up from $(t_0, 0)$ to $(t_0 + k, h)$.
- Then ramps down from $(t_0 + k, h)$ to $(t_0 + 2k, 0)$.

Notice that this is simply two line segments. We have pairs of points for each, so let's write the equation of each line segment:

- Slope: $\frac{h}{k}$, so: $y - 0 = \frac{h}{k}(t - t_0)$.
- Slope: $\frac{-h}{k}$, so: $y - 0 = -\frac{h}{k}(t - (t_0 + 2k))$

Finally, use the "On-Off" switch for each line segment. The first line segment comes on at time t_0, off at $t_0 + k$):

$$(u_{t_0}(t) - u_{t_0+k}(t))\left(\frac{h}{k}(t - t_0)\right)$$

We'll add in the second switch,

$$(u_{t_0+k}(t) - u_{t_0+2k}(t))\left(-\frac{h}{k}(t - t_0 - 2k)\right)$$

Add everything together. To get the answer in the text, factor the slope out and expand (for us, you can leave your answer in this form).

$$(u_{t_0}(t) - u_{t_0+k}(t))\left(\frac{h}{k}(t - t_0)\right) + (u_{t_0+k}(t) - u_{t_0+2k}(t))\left(-\frac{h}{k}(t - t_0 - 2k)\right)$$

Alternatively, you could also have written the second line segment using the first ordered pair.

16. It is a little heavy on the algebra- Let's see how we do. With zero IC's, the Laplace transform is simple to construct:

$$(s^2 + \frac{1}{4}s + 1)Y = \frac{k}{s}(\mathrm{e}^{-3/2s} - \mathrm{e}^{-5/2s})$$

so that

$$Y = \frac{k}{s(s^2 + \frac{1}{4}s + 1)}(\mathrm{e}^{-3/2s} - \mathrm{e}^{-5/2s})$$

The important thing to see in this is that the overall construction is of the type:

$$Y(s) = k(\mathrm{e}^{-as} - \mathrm{e}^{-bs})H(s)$$

so that if we find $h(t)$, then the solution is (factoring out k):

$$y(t) = k(u_a(t)h(t - a) - u_b(t)h(t - b))$$

Now we perform partial fractions to determine $h(t)$. It looks worse than it is:

$$H(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + (1/4)s + 1} \quad \Rightarrow \quad A(s^2 + (1/4)s + 1) + s(Bs + C) = 1$$

From setting $s = 0$, we get $A = 1$, and substituting that back in, we get:

$$s^2 + \frac{1}{4}s + 1 + Bs^2 + Cs = 1 \quad \Rightarrow \quad B + 1 = 0, \quad C + 1/4 = 0$$

so that $B = -1$ and $C = -1/4$. Now,

$$H(s) = \frac{1}{s} - \frac{s + 1/4}{s^2 + \frac{1}{4}s + 1}$$

We'll focus on that second term, since the first is easy. We need to complete the square to get the term to look like Table entries 9 and 10:

$$\frac{s + \frac{1}{4}}{s^2 + \frac{1}{4}s + 1} = \frac{s + \frac{1}{4}}{(s^2 + \frac{1}{4}s + \frac{1}{64}) + 1 - \frac{1}{64}} = \frac{s + \frac{1}{4}}{(s + \frac{1}{8})^2 + \frac{63}{64}} = \frac{s + \frac{1}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}} + \frac{\frac{1}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}}$$

The first term is ready for the Table. It's inverse transform is

$$\mathrm{e}^{-t/8} \cos\left(\sqrt{\frac{63}{64}}\, t\right) = \mathrm{e}^{-t/8} \cos\left(\frac{3\sqrt{7}}{8}\, t\right)$$

For the other term, we have:

$$\frac{1}{8} \cdot \frac{8}{3\sqrt{7}} \cdot \frac{\frac{3\sqrt{7}}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}} \quad \Rightarrow \quad \frac{1}{3\sqrt{7}} \mathrm{e}^{-t/8} \sin\left(\frac{3\sqrt{7}}{8}\, t\right)$$

so that

$$h(t) = 1 - \mathrm{e}^{-t/8} \cos\left(\frac{3\sqrt{7}}{8} t\right) - \frac{1}{3\sqrt{7}} \mathrm{e}^{-t/8} \sin\left(\frac{3\sqrt{7}}{8} t\right)$$

The overall solution is therefore: $ku_{3/2}(t)h(t - 3/2) - ku_{5/2}(t)h(t - 5/2)$.

For the rest, the software program Maple is required (optional). Here is the code to help with (d), and part (e) is similarly solved by using the graphics in Maple. Part (d) is answered via a small animation. (For fun, you can convert Maple animation into an (animated) GIF file to post on the web! To do this, right-click on the animation, choose Export, choose GIF, then save. To see the animation, open it using Firefox or any web browser).

```
Eqn16:=diff(y(t),t$2)+(1/4)*diff(y(t),t)+y(t)=k*Heaviside(t-3/2)-k*Heaviside(t-5/2);
Y1:=dsolve({Eqn16,y(0)=0,D(y)(0)=0},y(t));
animate(plot,[rhs(Y1)-2,t=0..9], k=2..3);
```

19. Straightforward to write down, a little tricky to analyze:

$$y'' + y = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)$$

Taking the Laplace transform, solve for Y and inverting:

$$(s^2+1)Y = \frac{1}{s} + 2\sum_{k=1}^{n}(-1)^k \mathrm{e}^{-k\pi s}\frac{1}{s} \quad \Rightarrow \quad Y = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^{n}(-1)^k \mathrm{e}^{-k\pi s}\frac{1}{s(s^2+1)}$$

where

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

so that

$$y(t) = 1 - \cos(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi t}\left(1 - \cos(t-k\pi)\right)$$

This is a bit difficult to analyze in this form. However, if we consider the graph of the cosine, we see that:

If k is odd	$\cos(t-k\pi) = -\cos(t)$
If k is even	$\cos(t-k\pi) = \cos(t)$

Here we write out the function in the form of a table:

kth term		Term is active:
$k=1$	$-2(1+\cos(t))$	$t \ge \pi$
$k=2$	$2(1-\cos(t))$	$t \ge 2\pi$
$k=3$	$-2(1+\cos(t))$	$t \ge 3\pi$
$k=4$	$2(1-\cos(t))$	$t \ge 4\pi$
$\vdots$	$\vdots$	

Therefore, writing $y(t)$ in piecewise form (for clarity, I'm writing it as a table):

Interval	$y(t)$
$t < \pi$	$-\cos(t) + 1$
$\pi \le t < 2\pi$	$(1-\cos(t) - 2(1+\cos(t)) = -3\cos(t) - 1$
$2\pi \le t < 3\pi$	$3(1-\cos(t)) - 2(1+\cos(t)) = -5\cos(t) + 1$
$3\pi \le t < 4\pi$	$3(1-\cos(t)) - 4(1+\cos(t)) = -7\cos(t) - 1$
$4\pi \le t < 5\pi$	$5(1-\cos(t)) - 4(1+\cos(t)) = -9\cos(t) + 1$

Here is the Maple code to help (we'll use the `sum` command!)

```
N:=15;
Eqn19:=diff(y(t),t$2)+y(t)=Heaviside(t)+2*sum((-1)^k*Heaviside(t-k*Pi),k=1..N);
Y1:=dsolve({Eqn19,y(0)=0,D(y)(0)=0},y(t),method=laplace);
plot(rhs(Y1),t=0..20*Pi);
N:=30;
Eqn19:=diff(y(t),t$2)+y(t)=Heaviside(t)+2*sum((-1)^k*Heaviside(t-k*Pi),k=1..N);
Y1:=dsolve({Eqn19,y(0)=0,D(y)(0)=0},y(t),method=laplace);
plot(rhs(Y1),t=0..50*Pi);
```

20. Keep the big picture in mind. In this case, when we invert the transform, we get:

$$Y(s) = \frac{1 + 2\sum(-1)^k \mathrm{e}^{-s\pi k}}{s(s^2 + \frac{1}{10}s + 1} = \left(\mathrm{e}^{-0s} + 2\sum(-1)^k \mathrm{e}^{-s\pi k}\right) H(s)$$

so if we find the inverse Laplace transform of $H(s)$, then the solution overall is easy to write down:

$$y(t) = u_0(t)h(t-0) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)h(t-k\pi)$$

To analyze this solution, we'll use the computer (optional from here). Before we get too far ahead, let's invert $H(s)$:

$$H(s) = \frac{1}{s(s^2 + \frac{1}{10}s + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \frac{1}{10}s + 1}$$

so that

$$A(s^2 + \frac{1}{10}s + 1) + s(Bs + C) = 1$$

Setting $s = 0$, we get $A = 1$. Substituting that in, we get

$$s^2 + \frac{1}{10}s + 1 + Bs^2 + Cs = 1$$

from which we get $B = -1$ and $C = -\frac{1}{10}$. So far then:

$$H(s) = \frac{1}{s} - \frac{s + \frac{1}{10}}{s^2 + \frac{1}{10}s + 1}$$

Inverting from here we use the same technique as before- Complete the square in the denominator to use Table Entries 9, 10 (so we focus on only the last term):

$$\frac{s + \frac{1}{10}}{s^2 + \frac{1}{10}s + \frac{1}{400} + \frac{399}{400}} = \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} + \frac{1}{20}\,\frac{20}{\sqrt{399}}\,\frac{\frac{\sqrt{399}}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}}$$

We're ready to put $h(t)$ together now:

$$h(t) = 1 - \mathrm{e}^{-t/20}\cos\left(\frac{\sqrt{399}}{20}t\right) - \frac{1}{\sqrt{399}}\mathrm{e}^{-t/20}\sin\left(\frac{\sqrt{399}}{20}t\right)$$

And we gave the overall solution as:

$$y(t) = h(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)h(t-k\pi)$$

From here, the exercise is optional since we need some good computer software. If you've had the Calc Lab, here is the Maple commands to get the solution and plot the result:

```
with(inttrans):
N:=5;
Eqn20:=diff(y(t),t$2)+(1/10)*diff(y(t),t)+y(t)=Heaviside(t)+
          2*Sum((-1)^k*Heaviside(t-k*Pi),k=1..N);
Y1:=laplace(Eqn20,t,s);
Y2:=subs(y(0)=0,D(y)(0)=0,Y1);
Y3:=solve(Y2,laplace(y(t),t,s));
Y4:=invlaplace(Y3,s,t);
plot(Y4,t=0..100);
```

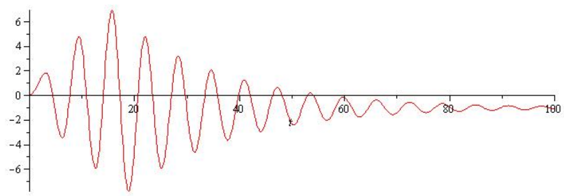

Figure 5: The plot of the solution to the DE in Exercise 20.

21. For the part we should be able to do by hand (as in 19, 20):

$$Y(s) = \frac{1 + \sum(-1)^k \mathrm{e}^{-k\pi s}}{s(s^2+1)}$$

Again this is of the form:

$$Y(s) = (\mathrm{e}^{-0s} + \sum(-1)^k \mathrm{e}^{-k\pi s})H(s)$$

so if we can find $h(t)$, then the solution will be:

$$y(t) = h(t) + \sum_{k=1}^{n} (-1)^k u_{k\pi}(t) h(t - k\pi)$$

and then we'll plot it using the computer software. In this case,

$$H(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1} = \frac{1}{s} + \frac{s}{s^2+1} \quad \Rightarrow \quad h(t) = 1 + \cos(t)$$

The plot is given in the figure below for $n = 15$, as asked for in the problem.

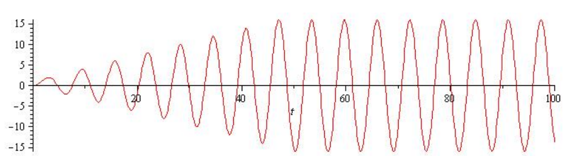

Figure 6: The plot of the solution to the DE in Exercise 21.

22. Similarly, we can do the same for 22: The only thing that changes is $H(s)$, since

$$Y(s) = \frac{1 + \sum(-1)^k \mathrm{e}^{-k\pi s}}{s(s^2 + \frac{1}{10}s + 1)}$$

In this case,

$$H(s) = \frac{1}{s(s^2 + \frac{1}{10}s + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \frac{1}{10}s + 1} \quad \Rightarrow \quad A = 1, \quad B = -1, \quad C = -\frac{1}{10}$$

Just like last time, the second term is the more difficult to invert, but luckily we already did it in Exercise 20 (See above):

$$\frac{s + \frac{1}{10}}{s^2 + \frac{1}{10}s + \frac{1}{400} + \frac{399}{400}} = \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} + \frac{1}{20} \frac{20}{\sqrt{399}} \frac{\frac{\sqrt{399}}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}}$$

so the $h(t)$ is the same as well.

For the rest (optional), we'll plot using the software, and the result is shown below.

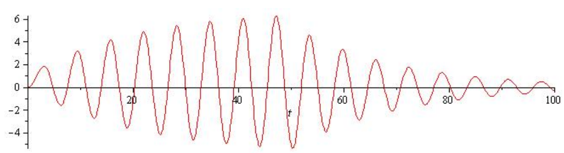

Figure 7: The plot of the solution to the DE in Exercise 22.