Math 244
Homework To Replace 7.2

1. Exercise 22 (Section 7.1, p 363) SOLUTION: Tanks, as in class.

$$\begin{aligned} Q_1' &= \tfrac{3}{2} - \tfrac{1}{10}Q_1 + \tfrac{3}{40}Q_2 \\ Q_2' &= 3 + \tfrac{3}{10}Q_1 - \tfrac{1}{5}Q_2 \end{aligned} \qquad Q_1(0) = 25, \quad Q_2(0) = 15$$

2. Give the solution to each system. If it has an infinite number of solutions, give your answer in vector form:

   SOLUTION: For the first one, there is only one solution. Find it any way you want- For review, here is Cramer's Rule:

$$\begin{aligned} 3x + 2y &= 1 \\ 2x - y &= 3 \end{aligned} \qquad x = \frac{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = 1 \quad y = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = -1$$

   For the second one, there is no solution (parallel lines).

$$\begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= 3 \end{aligned}$$

   For the third problem, we have the same line: $3x + 2y = 1$. To make this vector form, assume one of the variables is free (let's take y to be free), then write x and y:

$$\begin{aligned} x &= \tfrac{1}{3} - \tfrac{2}{3}y \\ y &= \qquad y \end{aligned} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$$

3. Write each of the previous systems in matrix-vector form. Verify that the determinant of the first matrix is not zero, but is zero for the second and third.

   SOLUTION:

$$\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4. Write each system of differential equations in matrix-vector form or write the system from the matrix-vector form:

$$\begin{aligned} x_1' &= 3x_1 - x_2 \\ x_2' &= 9x_1 - 3x_2 \end{aligned} \qquad \mathbf{x}' = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

   SOLUTION:

$$\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \mathbf{x} \qquad \begin{aligned} x_1' &= -x_1 + 2x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned}$$

5. Find the equilibrium solutions to the previous autonomous linear differential equations.

   SOLUTION: An equilibrium solution is found by setting the derivative to zero. For the first system, we see that there are an infinite number of solutions, and we write the solution in parametric vector form.

$$\begin{aligned} 3x_1 - x_2 &= 0 \\ 9x_2 - 3x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= x_1 \\ x_2 &= 3x_1 \end{aligned} \qquad \Rightarrow \qquad \mathbf{x} = k \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

   In the second equation, $(0, 0)$ is the only solution.

For the following problems, define

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

6. Similar to Exercise 1 (Section 7.2). Given the matrices A and B above:

7. $A - 4B = \begin{bmatrix} -11 & -2 \\ -5 & -11 \end{bmatrix}$

8. $(B - 4I)\mathbf{b} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

9. $\det(B - 4I) = 0$

10. $AB = \begin{bmatrix} 5 & 7 \\ -2 & 2 \end{bmatrix}$, $BA \begin{bmatrix} 2 & 7 \\ -2 & 5 \end{bmatrix}$

11. Compute $A^{-1}\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

12. Verify: $A\mathbf{b} - 3\mathbf{b} = (A - 3I)\mathbf{b}$

13. $B^T B = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$, $\text{Tr}(A) = 2$, $\text{Tr}(B) = 8$

14. Is $A^T = A$? No. Is $B^T = B$? Yes

15. Solve the system of complex equations. Before we start, check that we get the bottom equation if we divide the top equation by $(1 + i)$.

$$\begin{array}{rrl} -(1+i)v_1 & +2v_2 & = 0 \\ -v_1 & +(1-i)v_2 & = 0 \end{array}$$

SOLUTION:

For the check, we want to multiply top and bottom by the conjugate so that we're not dividing by a complex number:

$$\frac{2}{1+i} = \frac{2(1-i)}{(1+i)(1-i)} = \frac{2(1-i)}{2} = 1 - i$$

That is what we expected- Now we only need either the top equation or the bottom equation. Using the bottom equation, we'll let v_2 be the free variable, and write the solution in parametric vector form:

$$\begin{array}{ll} v_1 & = (1-i)v_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = v_2 \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

16. Solve the system of complex equations. Before we start, check that we get the bottom equation if we divide the top equation by $(2 - i)$.

$$\begin{array}{rrl} (2-i)v_1 & -5v_2 & = 0 \\ v_1 & -(2+i)v_2 & = 0 \end{array}$$

SOLUTION:

For the check, we want to multiply top and bottom by the conjugate so that we're not dividing by a complex number:

$$\frac{-5}{2-i} = \frac{-5(2+i)}{(2+i)(2-i)} = \frac{-5(2+i)}{5} = -(2+i)$$

That means that the two equations are constant multiples of each other, and we only need either the top equation or the bottom equation. Using the bottom equation again, we'll let v_2 be the free variable, and write the solution in parametric vector form:

$$\begin{array}{ll} v_1 & = (2+i)v_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = v_2 \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

17. Exercises 22, 23 (Section 7.2, p. 373)

(a) If $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$, verify that a solution is $\mathbf{x}(t) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t}$

SOLUTION: To verify, we'll first compute $\mathbf{x}'$, then compare it to $A\mathbf{x}$. First, the derivative- t only appears in the function out front:

$$\mathbf{x}' = 2e^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

On the other hand, now we'll compute $A\mathbf{x}$. Treat e^{2t} as a constant:

$$A\mathbf{x} = e^{2t} \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = e^{2t} \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

Comparing the two expressions, we see that $\mathbf{x}'$ and $A\mathbf{x}$ are the same.

(b) If $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$, verify that a solution is $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^{2t}$

TYPO: (See p. 373, #23 The last term, e^{2t} should be e^t (Sorry!)

SOLUTION: This one is done much the same way. First, compute the derivative, then we'll compare it to the other expression. When we differentiate, remember to use the product rule.

$$\mathbf{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} (e^t + te^t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t$$

Now we'll do the matrix-vector multiplication from the right side of the DE:

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t =$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t =$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t$$

And again we see that the two expressions are the same.