Solve the linear first order system $\mathbf{x}' = A\mathbf{x}$

Given

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x_1' \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}$$

Then we could solve the system in three ways:

- Convert the system to a single second order DE and use Chapter 3 methods. Note that we should also be able to go backwards- Given a second order DE, convert it into a system of first order. There are some questions in the review that will extend this to nonlinear DEs and perhaps third order.
- Could try to write the system in implicit form:

  $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{cx_1 + dx_2}{ax_1 + bx_2}$$

  Then we could try to use a method from Chapter 2. We can also do this with many nonlinear systems of equations (there are a couple of them in the review).
- Use the eigenvalues and eigenvectors from the matrix A. This is the topic of Ch 7.

Eigenvalues and Eigenvectors

- Definition: Given an $n \times n$ matrix A, if there is a constant λ and a non-zero vector $\mathbf{v}$ so that

  $$\begin{aligned} av_1 + bv_2 &= \lambda v_1 \\ cv_1 + dv_2 &= \lambda v_2 \end{aligned} \qquad \text{or} \qquad A\mathbf{v} = \lambda\mathbf{v}$$

  then λ is an eigenvalue, and $\mathbf{v}$ is an associated eigenvector for matrix A. Note that an eigenvector is not uniquely determined; we usually choose the simplest vector (integer valued if possible).
- If we try to solve our equation:

  $$\begin{array}{rrl} av_1 & +bv_2 & = \lambda v_1 \\ cv_1 & +dv_2 & = \lambda v_2 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array} \qquad \text{or} \qquad (A - \lambda I)\mathbf{v} = \mathbf{0} \tag{1}$$

  This system has a non-trivial (non-zero) solution for v_1, v_2 only if the determinant of coefficients is 0:

  $$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$$

  And this is the **characteristic equation**. We solve this for the eigenvalues:

  $$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \quad \Leftrightarrow \lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$$

  where $\text{Tr}(A)$ is the trace of A (which we defined as $a + d$). For each λ, we must go back and solve Equation (1).

- Given a λ and $\mathbf{v}$, the generalized eigenvector $\mathbf{w}$ is computed as below, and is used when we are solving the associated system of differential equations.

$$\begin{aligned} (a-\lambda)w_1 + bw_2 &= v_1 \\ cw_1 + (d-\lambda)w_2 &= v_2 \end{aligned} \tag{2}$$

Summary

To solve $\mathbf{x}' = A\mathbf{x}$, find the trace, determinant and discriminant for the matrix A. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 give two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \mathrm{Real}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Imag}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector $\mathbf{v}$. Compute the generalized eigenvector $\mathbf{w}$ using Equation 2. Then

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(C_1\mathbf{v} + C_2\left(t\mathbf{v} + \mathbf{w}\right)\right)$$

You might find this helpful- Below there is a chart comparing the solutions from Chapter 3 to the solutions in Chapter 7:

	Chapter 3	Chapter 7
Form:	$ay'' + by' + cy = 0$	$\mathbf{x}' = A\mathbf{x}$
Ansatz:	$y = \mathrm{e}^{rt}$	$\mathbf{x} = \mathrm{e}^{\lambda t}\mathbf{v}$
Char Eqn:	$ar^2 + br + c = 0$	$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$
Real Solns	$y = C_1\mathrm{e}^{r_1 t} + C_2\mathrm{e}^{r_2 t}$	$\mathbf{x}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$
Complex	$y = C_1\mathrm{Re}(\mathrm{e}^{rt}) + C_2\mathrm{Im}(\mathrm{e}^{rt})$	$\mathbf{x}(t) = C_1\mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2\mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$
Repeated Root	$y = \mathrm{e}^{rt}(C_1 + C_2 t)$	$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(C_1\mathbf{v} + C_2\left(t\mathbf{v} + \mathbf{w}\right)\right)$

Poincaré Diagram

Because we have the trace, determinant and discriminant, we can classify the origin as to its stability using the Poincaré diagram (which will be provided).