Exercise Set 3 (HW to replace 7.3, 7.5)

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

2. For each matrix, find the eigenvalues and eigenvectors (these are selected from 16-23, p. 384 in the textbook). Note that they could be complex, and the matrix A may have complex numbers. Try the last one to see if you can do it!

   (a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

   (f) $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$

3. For each given λ and $\mathbf{v}$, find an expression for $\mathrm{Re}(e^{\lambda t}\mathbf{v})$ and $\mathrm{Im}(e^{\lambda t}\mathbf{v})$:

   (a) $\lambda = 3i$, $\mathbf{v} = [1 - i, 2i]^T$

   (b) $\lambda = 1 + i$, $\mathbf{v} = [i, 2]^T$

   (c) $\lambda = 2 - i$, $\mathbf{v} = [1, 1 + 2i]^T$

   (d) $\lambda = i$, $\mathbf{v} = [2 + 3i, 1 + i]^T$

4. Given the eigenvalues and eigenvectors for some matrix A, write the general solution to $\mathbf{x}' = A\mathbf{x}$. Furthermore, classify the origin as a sink, source, saddle, or none of the above.

   (a) $\lambda = -2, 3 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (b) $\lambda = -2, -2 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

   (c) $\lambda = 2, -3 \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

5. Give the general solution to each system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, and sketch a phase plane (solutions in the x_1, x_2 plane). Identify the origin as a *sink*, *source* or *saddle*:

   (a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -6 & 10 \\ -2 & 3 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 8 & 6 \\ -15 & -11 \end{bmatrix}$

6. (Extra Practice) For each system below, find y as a function of x by first writing the differential equation as dy/dx.

   (a) $\begin{aligned} x' &= -2x \\ y' &= y \end{aligned}$

   (b) $\begin{aligned} x' &= y + x^3 y \\ y' &= x^2 \end{aligned}$

   (c) $\begin{aligned} x' &= -(2x + 3) \\ y' &= 2y - 2 \end{aligned}$

   (d) $\begin{aligned} x' &= -2y \\ y' &= 2x \end{aligned}$