Exercise Set 3 (HW to replace 7.3, 7.5)

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

SOLUTION: On the one hand, if we differentiate, we get:

$$\mathbf{x}'(t) = -C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

On the other hand, if we look at $A\mathbf{x}$:

$$C_1 \mathrm{e}^{-t} \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = C_1 \mathrm{e}^{-t} \begin{bmatrix} -1 \\ -2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Comparing these expressions, we see they are the same.

2. For each matrix, find the eigenvalues and eigenvectors (these are selected from 16-23, p. 384 in the textbook). Note that they could be complex, and the matrix A may have complex numbers. Try the last one to see if you can do it!

(a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$ $\quad$ $\lambda^2 - 6\lambda + 8 = 0$, $(\lambda - 2)(\lambda - 4) = 0$, $\lambda = 2, \quad \lambda = 4$ $\quad$ For $\lambda = 2$: $3v_1 - v_2 = 0$, $\mathbf{v} = [1, 3]^T$ $\quad$ For $\lambda = 4$: $v_1 - v_2 = 0$, $\mathbf{v} = [1, 1]^T$

(b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ $\quad$ $\lambda^2 - 2\lambda + 5 = 0$, $(\lambda - 1)^2 + 4 = 0$, $\lambda = 1 \pm 2i$ $\quad$ For $\lambda = 1 + 2i$: $(2 + 2i)v_1 - 2v_2 = 0$ $\quad$ $\mathbf{v} = [1, 1 - i]^T$ or $\mathbf{v} = [1 + i, 2]^T$

(c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ $\quad$ $\lambda^2 - 4\lambda + 3 = 0$, $(\lambda + 1)(\lambda + 3) = 0$, $\lambda = -1, \quad \lambda = -3$ $\quad$ For $\lambda = -1$: $-v_1 + v_2 = 0$, $\mathbf{v} = [1, 1]^T$ $\quad$ For $\lambda = -3$: $v_1 + v_2 = 0$, $\mathbf{v} = [-1, 1]^T$

(d) $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ $\quad$ $\lambda^2 - 2\lambda = 0$, $\lambda = 0, \quad \lambda = 2$ $\quad$ For $\lambda = 0$: $v_1 + iv_2 = 0$, $\mathbf{v} = [-i, 1]^T$ $\quad$ For $\lambda = 2$: $-v_1 + iv_2 = 0$, $\mathbf{v} = [i, 1]^T$

(e) $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ $\quad$ $\lambda^2 - 4 = 0$, $\lambda = \pm 2$ $\quad$ For $\lambda = 2$: $-v_1 + \sqrt{3}v_2 = 0$, $\mathbf{v} = [\sqrt{3}, 1]^T$ $\quad$ For $\lambda = -2$: $3v_1 + \sqrt{3}v_2 = 0$, $\mathbf{v} = [-\sqrt{3}/3, 1]^T$

(f) (Skip this unless you've had Math 240)

3. For each given λ and $\mathbf{v}$, find an expression for $\mathrm{Re}(\mathrm{e}^{\lambda t}\mathbf{v})$ and $\mathrm{Im}(\mathrm{e}^{\lambda t}\mathbf{v})$:

(a) $\lambda = 3i$, $\mathbf{v} = [1 - i, 2i]^T$

SOLUTION: $\mathrm{e}^{3it} = \cos(3t) + i\sin(3t)$, so we multiply the vector by this:

$$(\cos(3t) + i\sin(3t)) \begin{bmatrix} 1 - i \\ 2i \end{bmatrix} = \begin{bmatrix} (\cos(3t) + \sin(3t)) + i(\sin(3t) - \cos(3t)) \\ -2\sin(3t) + 2i\cos(3t) \end{bmatrix}$$

So now write the real and imaginary parts separately:

$$\mathrm{Re}(\mathrm{e}^{\lambda t}\mathbf{v}) = \begin{bmatrix} \cos(3t) + \sin(3t) \\ -2\sin(3t) \end{bmatrix} \qquad \mathrm{Im}(\mathrm{e}^{\lambda t}\mathbf{v}) = \begin{bmatrix} \sin(3t) - \cos(3t) \\ 2\cos(3t) \end{bmatrix}$$

(b) $\lambda = 1 + i$, $\mathbf{v} = [i, 2]^T$

SOLUTION: $\mathrm{e}^{(1+i)t} = \mathrm{e}^t \mathrm{e}^{it} = \mathrm{e}^t(\cos(t) + i\sin(t))$, so we multiply the vector by this (keep the exponential out)

$$\mathrm{e}^t(\cos(t) + i\sin(t)) \begin{bmatrix} i \\ 2 \end{bmatrix} = \begin{bmatrix} -\sin(t) + i\cos(t)) \\ 2\cos(t) + i2\sin(t) \end{bmatrix}$$

So now write the real and imaginary parts separately:

$$\mathrm{Re}(\mathrm{e}^{\lambda t}\mathbf{v}) = \mathrm{e}^t \begin{bmatrix} -\sin(t) \\ 2\cos(3t) \end{bmatrix} \qquad \mathrm{Im}(\mathrm{e}^{\lambda t}\mathbf{v}) = \mathrm{e}^t \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$$

(c) $\lambda = 2 - i$, $\mathbf{v} = [1, 1 + 2i]^T$

SOLUTION: $\mathrm{e}^{(2-i)t} = \mathrm{e}^{2t}(\cos(-t) + i\sin(-t)) = \mathrm{e}^{2t}(\cos(t) - i\sin(t))$, (cosine is even, sine is odd).

$$\mathrm{e}^{2t}(\cos(t) - i\sin(t)) \begin{bmatrix} 1 \\ 1 + 2i \end{bmatrix} = \begin{bmatrix} \cos(t) - i\sin(t) \\ (\cos(t) + 2\sin(t)) + i(2\cos(t) - \sin(t)) \end{bmatrix}$$

So now write the real and imaginary parts separately:

$$\mathrm{Re}(\mathrm{e}^{\lambda t}\mathbf{v}) = \mathrm{e}^{2t} \begin{bmatrix} \cos(t) \\ \cos(t) + 2\sin(t) \end{bmatrix} \qquad \mathrm{Im}(\mathrm{e}^{\lambda t}\mathbf{v}) = \mathrm{e}^{2t} \begin{bmatrix} -\sin(t) \\ 2\cos(t) - \sin(t) \end{bmatrix}$$

(d) $\lambda = i$, $\mathbf{v} = [2 + 3i, 1 + i]^T$

SOLUTION: $\mathrm{e}^{it} = \cos(t) + i\sin(t)$, so we multiply the vector by this:

$$(\cos(t) + i\sin(t)) \begin{bmatrix} 2 + 3i \\ 1 + i \end{bmatrix} = \begin{bmatrix} (2\cos(t) - 3\sin(t)) + i(2\sin(t) + 3\cos(t)) \\ (\cos(t) - \sin(t)) + i(\cos(t) + \sin(t)) \end{bmatrix}$$

So now write the real and imaginary parts separately:

$$\mathrm{Re}(\mathrm{e}^{\lambda t}\mathbf{v}) = \begin{bmatrix} 2\cos(t) - 3\sin(t) \\ \cos(t) - \sin(t) \end{bmatrix} \qquad \mathrm{Im}(\mathrm{e}^{\lambda t}\mathbf{v}) = \begin{bmatrix} 2\sin(t) + 3\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix}$$

4. Given the eigenvalues and eigenvectors for some matrix A, write the general solution to $\mathbf{x}' = A\mathbf{x}$. Furthermore, classify the origin as a sink, source, saddle, or none of the above.

(a) $\lambda = -2, 3 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

SOLUTION:

$$\mathbf{x} = C_1\mathrm{e}^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2\mathrm{e}^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

And the origin is a saddle.

(b) $\lambda = -2, -2 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

TYPO: We just barely covered this case last time- Not on the quiz (This is the case where we have a repeated root with only one eigenvector)

(c) $\lambda = 2, -3 \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

SOLUTION:

$$\mathbf{x} = C_1\mathrm{e}^{2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2\mathrm{e}^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

And the origin is a saddle.

5. Give the general solution to each system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, and sketch a phase plane (solutions in the x_1, x_2 plane). Identify the origin as a *sink*, *source* or *saddle*:

   (a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$
   SOLUTION: We compute the eigenvalues and eigenvectors first. The final answer is

$$\mathbf{x}(t) = C_1 e^{-4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   (The origin is a saddle).

   (b) $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$
   SOLUTION: We compute the eigenvalues and eigenvectors first. The final answer is

$$\mathbf{x}(t) = C_1 e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

   The origin is a source.

   (c) $A = \begin{bmatrix} -6 & 10 \\ -2 & 3 \end{bmatrix}$
   SOLUTION: We compute the eigenvalues and eigenvectors first. The final answer is

$$\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   The origin is a sink.

   (d) $A = \begin{bmatrix} 8 & 6 \\ -15 & -11 \end{bmatrix}$
   SOLUTION: We compute the eigenvalues and eigenvectors first. The final answer is

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

   The origin is a sink.

6. (Extra Practice) For each system below, find y as a function of x by first writing the differential equation as dy/dx.

   (a) $\begin{aligned} x' &= -2x \\ y' &= y \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{-2x} \quad \Rightarrow \quad \frac{1}{y}\, dy = -\frac{1}{2}\frac{1}{x}\, dx \quad \Rightarrow \quad \ln|y| = -\frac{1}{2}\ln|x| + C$
   so $y = \frac{A}{\sqrt{x}}$

   (b) $\begin{aligned} x' &= y + x^3 y \\ y' &= x^2 \end{aligned} \quad \Rightarrow \quad y\, dy = \frac{x^2}{1+x^3}\, dx \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{2}\ln|1+x^3| + C$

   (c) $\begin{aligned} x' &= -(2x+3) \\ y' &= 2y - 2 \end{aligned} \quad \Rightarrow \quad -(2y-1) - (2x+3)\frac{dy}{dx} = 0$
   This is exact: $2x - 3y - 2xy = C$.

   (d) $\begin{aligned} x' &= -2y \\ y' &= 2x \end{aligned} \quad \Rightarrow \quad x^2 + y^2 = C$