Exercise Set 4 (HW for 7.6, 7.8)

1. Solve $\mathbf{x}' = A\mathbf{x}$, where A is given below:

(a) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ $\qquad \begin{array}{c} \text{Tr}(A) = 2 \\ \det(A) = 5 \\ \Delta = 4 - 20 = -16 \end{array} \qquad \begin{array}{c} \lambda = 1 + 2i \\ \text{Spiral Source} \\ \mathbf{v} = [1+i, 2]^T \end{array}$

The solution will be $C_1 \text{Re}(e^{\lambda t}\mathbf{v}) + C_2 \text{Im}(e^{\lambda t}\mathbf{v})$

$$\mathbf{x}(t) = e^t \left[C_1 \begin{bmatrix} \cos(2t) - \sin(2t) \\ 2\cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} \cos(2t) + \sin(2t) \\ 2\sin(2t) \end{bmatrix} \right]$$

(b) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ $\qquad \begin{array}{c} \text{Tr}(A) = 2 \\ \det(A) = 1 \\ \Delta = 4 - 4 = 0 \end{array} \qquad \begin{array}{c} \lambda = 1, 1 \\ \text{Degen Source} \\ \mathbf{v} = [2, 1]^T \end{array}$

We need to compute the generalized eigenvector $\mathbf{w}$ that satisfies the equation:

$$\begin{array}{rl} (3-1)w_1 - 4w_2 & = 2 \\ w_1 + (-1-1)w_2 & = 1 \end{array} \quad \Rightarrow \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The solution is therefore $\mathbf{x}(t) = e^{\lambda t}(C_1\mathbf{v} + C_2(t\mathbf{v} + \mathbf{w}))$, which is

$$\mathbf{x}(t) = e^t \left(C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right)$$

(c) $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$ $\qquad \begin{array}{c} \text{Tr}(A) = 0 \\ \det(A) = 1 \\ \Delta = -4 \end{array} \qquad \begin{array}{c} \lambda = i \\ \text{Center} \\ \mathbf{v} = [5, 2-i]^T \end{array}$

The solution will be $C_1 \text{Re}(e^{\lambda t}\mathbf{v}) + C_2 \text{Im}(e^{\lambda t}\mathbf{v})$

$$\mathbf{x}(t) = \left[C_1 \begin{bmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + C_2 \begin{bmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix} \right]$$

(d) $A = \begin{bmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{bmatrix}$ $\qquad \begin{array}{c} \text{Tr}(A) = -2 \\ \det(A) = 1 \\ \Delta = 4 - 4 = 0 \end{array} \qquad \begin{array}{c} \lambda = -1, -1 \\ \text{Degen Sink} \\ \mathbf{v} = [2, 1]^T \end{array}$

We need to compute the generalized eigenvector $\mathbf{w}$ that satisfies the equation:

$$\begin{array}{rl} (-3/2+1)w_1 + w_2 & = 2 \\ (-1/4)w_1 + (-1/2+1)w_2 & = 1 \end{array} \quad \Rightarrow \quad \mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The solution is therefore $\mathbf{x}(t) = e^{\lambda t}(C_1\mathbf{v} + C_2(t\mathbf{v} + \mathbf{w}))$, which is

$$\mathbf{x}(t) = e^{-t} \left(C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \right)$$

(e) $A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$ $\quad \begin{array}{c} \mathrm{Tr}(A) = -2 \\ \det(A) = 2 \\ \Delta = 4 - 8 = -4 \end{array} \quad \begin{array}{c} \lambda = -1 + i \\ \text{Spiral Sink} \\ \mathbf{v} = [2 + i, 5]^T \end{array}$

The solution will be $C_1 \mathrm{Re}(\mathrm{e}^{\lambda t} \mathbf{v}) + C_2 \mathrm{Im}(\mathrm{e}^{\lambda t} \mathbf{v})$

$$\mathbf{x}(t) = \mathrm{e}^{-t} \left[C_1 \begin{bmatrix} 2\cos(t) - \sin(t) \\ 5\cos(t) \end{bmatrix} + C_2 \begin{bmatrix} 2\sin(t) + \cos(t) \\ 5\sin(t) \end{bmatrix} \right]$$

2. For 1(a) and 1(b), convert to a second order differential equation and solve by methods from Chapter 3.

SOLUTION: Looking at the solution in 1(a), I should probably solve for the differential equation in x_2. However, in this case, we'll solve for x_1 and then just point to how we would compute x_2- It will not be in exactly the same form as our solution to 1(a), but it will be equivalent.

For 1(a), solving the first equation for x_2, we get

$$x_2 = -\frac{1}{2}(x_1' - 3x_1)$$

Substitution into the second equation gives us:

$$-\frac{1}{2}(x_1'' - 3x_1') = 4x_1 + \frac{1}{2}(x_1' - 3x_1)$$

Multiply both sides by -2 and simplify to get:

$$x_1'' - 2x_1' + 5x_1 == 0 \quad \Rightarrow \quad r^2 - 2r + 5 = 0 \quad \Rightarrow \quad r = 1 \pm 2i$$

Therefore, $x_1(t) = \mathrm{e}^t(C_1 \cos(2t) + C_2 \sin(2t))$. To find x_2, substitute x_1 into the equation above- This is a bit of a pain, which is why we are computing these solutions using eigenvalues and eigenvectors.

For 1(b), I will try solving for the other function first. Then: $x_1 = x_2' + x_2$, and substitution into the first equation:

$$x_2'' + x_2' = 3(x_2' + x_2) - 4x_2 \quad \Rightarrow \quad x_2'' - 2x_2' + x_2 = 0$$

The characteristic polynomial is $r^2 - 2r + 1 = 0$, so we get $r = 1, 1$ and

$$x_2(t) = \mathrm{e}^t(C_1 + C_2 t)$$

To find x_1, we compute $x_2' + x_2$ to get:

$$x_1 = x_2' + x_2 = \mathrm{e}^t(2C_1 + C_2(2t + 1))$$

and you can verify that this is what we had before in 1(b).

3. Given the eigenvalues and eigenvectors for some matrix A, write the general solution to $\mathbf{x}' = A\mathbf{x}$. Furthermore, classify the origin as a sink, source, spiral sink, spiral source, saddle, or none of the above.

(a) $\lambda = -1 + 2i \qquad \mathbf{v} = \begin{bmatrix} 1 - i \\ 2 \end{bmatrix}$

$$\mathbf{x}(t) = \mathrm{e}^{-t}\left(C_1 \begin{bmatrix} \cos(2t) + \sin(2t) \\ 2\cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2t) - \cos(2t) \\ 2\sin(2t) \end{bmatrix} \right)$$

(Spiral Sink)

(b) $\lambda = -2, 3 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\mathbf{x}(t) = C_1\mathrm{e}^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2\mathrm{e}^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(Saddle)

(c) $\lambda = -2, -2 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\mathbf{x}(t) = \mathrm{e}^{-2t}\left(C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \right)$$

(Degenerate Sink)

(d) $\lambda = 2, -3 \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\mathbf{x}(t) = C_1\mathrm{e}^{2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2\mathrm{e}^{-3t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(Saddle)

(e) $\lambda = 1 + 3i \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$

$$\mathbf{x}(t) = \mathrm{e}^{t}\left(C_1 \begin{bmatrix} \cos(3t) \\ \cos(3t) + \sin(3t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(3t) \\ \sin(3t) - \cos(3t) \end{bmatrix} \right)$$

(Spiral Source)

(f) $\lambda = 2i \qquad \mathbf{v} = \begin{bmatrix} 1 + i \\ 1 \end{bmatrix}$

$$\mathbf{x}(t) = \left(C_1 \begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2t) + \cos(2t) \\ \sin(2t) \end{bmatrix} \right)$$

(Center)

4. (Also see 13-20, p. 410) For each system $\mathbf{x}' = A\mathbf{x}$, the matrix A will depend upon the parameter α: (i) Determine the eigenvalues in terms of α, (ii) Find the critical values of α where the behavior of the solution to the system changes significantly. We'll go through one or two in class.

   (a) $\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$
   $$\begin{gathered} \text{Tr}(A) = 2\alpha \\ \det(A) = \alpha^2 + 1 \\ \Delta = (2\alpha)^2 - 4(\alpha^2+1) = -4 \end{gathered}$$

   In the Poincaré diagram, the determinant is positive and the discriminant is negative puts us "inside" the parabola. Therefore:

   - If $\alpha < 0$, the origin is a spiral sink.
   - If $\alpha = 0$, the origin is a center.
   - If $\alpha > 0$, the origin is a spiral source.

   (b) $\begin{bmatrix} 0 & -5 \\ 1 & \alpha \end{bmatrix}$
   $$\begin{gathered} \text{Tr}(A) = \alpha \\ \det(A) = 5 \\ \Delta = \alpha^2 - 20 \end{gathered}$$

   In the Poincaré diagram, you can think of us being on the horizontal line at $\det(A) = 5$. We see that $\Delta = 0$ when $\alpha = \pm 2\sqrt{5}$, which tells us when we are inside or outside the parabola:

   - If $\alpha < -2\sqrt{5}$, the origin is a sink.
   - If $\alpha = -2\sqrt{5}$, the origin is a degenerate sink.
   - If $-2\sqrt{5} < \alpha < 0$, the origin is a spiral sink.
   - If $\alpha = 0$, the origin is a center.
   - If $0 < \alpha < 2\sqrt{5}$, the origin is a spiral source.
   - If $\alpha = 2\sqrt{5}$, the origin is a degenerate source.
   - If $\alpha > 2\sqrt{5}$, the origin is a source.

   (c) $\begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$
   $$\begin{gathered} \text{Tr}(A) = 0 \\ \det(A) = -4 + 5\alpha \\ \Delta = -4(-4+5\alpha) \end{gathered}$$

   On the diagram, we are on the vertical axis, so the classification depends only on whether the determinant is positive or negative:

   If $\alpha > 4/5$, then we have a center. If $\alpha = 4/5$, we have "uniform motion" (we have a double zero eigenvalue), and if $\alpha < 4/5$, we have a saddle.

   (d) $\begin{bmatrix} 3 & \alpha \\ -6 & -4 \end{bmatrix}$ $\begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$
   $$\begin{gathered} \text{Tr}(A) = -1 \\ \det(A) = 6(\alpha - 2) \\ \Delta = 49 - 24\alpha \end{gathered}$$

   We are on a vertical line where the trace is -1, and we have two critical points for α ($\alpha = 2$ and $\alpha = 49/24 \approx 2.04$).

   - For $\alpha < 2$, the determinant is negative and the origin is a saddle

- For $\alpha = 2$, we have a "line of stable fixed points"
- For $2 < \alpha < 2.04$, we have a sink.
- For $\alpha = 2.04$, we have a degenerate sink.
- For $\alpha > 2.04$, we have a spiral sink.