# Solutions, Exercise Set 5 (Finishing Chapter 7)

1. Solve $\mathbf{x}' = A\mathbf{x}$, where $A$ is given below. Also, classify the origin using the Poincaré diagram.

   (a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$ $\quad \begin{matrix} \text{Tr}(A) = 6 \\ \det(A) = 8 \\ \Delta = 4 \end{matrix}$

   SOLUTION: We can compute the trace, determinant and discriminant first- That will help us verify our later work and tell us what the origin is. In this case, we have a source, which means we'll have a two positive real eigenvalues:

   $$\lambda^2 - 6\lambda + 8 = 0 \quad \Rightarrow \quad (\lambda - 4)(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda = 2, 4$$

   For $\lambda_1 = 2$, the equation we solve is: $(5-2)v_1 - v_2 = 0$, or $v_2 = 3v_1$. Writing this in vector form, $\mathbf{v}_1 = [1, 3]^T$.
   For $\lambda_2 = 4$, the equation we solve is $(5-4)v_1 - v_2 = 0$, or $v_1 = v_2$. In vector form, $\mathbf{v} = [1, 1]^T$.
   In summary, the solution is:

   $$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   (b) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ $\quad \begin{matrix} \text{Tr}(A) = 2 \\ \det(A) = 1 \\ \Delta = 0 \end{matrix}$

   SOLUTION: Using the trace, determinant and discriminant, we see that we have a degenerate source- Meaning we have a double, positive, real eigenvalue:

   $$\lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad (\lambda - 1)^2 = 0 \quad \Rightarrow \quad \lambda = 1, 1$$

   For $\lambda = 1$, the equation we solve is: $(3-1)v_1 - 4v_2 = 0$, or $v_1 = 2v_2$. Writing this in vector form, $\mathbf{v} = [2, 1]^T$.
   For the generalized eigenvector $\mathbf{w}$, the equation we solve is $(3-1)w_1 - 4w_2 = 2$. A nice choice might be $\mathbf{w} = [1, 0]^T$.
   In summary, the solution is:

   $$\mathbf{x}(t) = C_1 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{t} \left( t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

   (c) $A = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$ $\quad \begin{matrix} \text{Tr}(A) = 0 \\ \det(A) = 0 \\ \Delta = 0 \end{matrix}$

SOLUTION: Using the trace, determinant and discriminant, we see that we have "uniform motion", or a double zero eigenvalue $\lambda = 0, 0$.

For an eigenvector, the equation we solve is: $4v_1 - 2v_2 = 0$, or $v_2 = 2v_1$. Writing this in vector form, $\mathbf{v} = [1, 2]^T$.

For the generalized eigenvector $\mathbf{w}$, the equation we solve is $4w_1 - 2w_2 = 1$. A nice choice might be $\mathbf{w} = [0, 1/2]^T$.

In summary, the solution is:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \left( t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \right) = \begin{bmatrix} c_1 + tc_2 \\ 2c_1 + 2tc_2 + c_2/2 \end{bmatrix}$$

(d) $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$ $\quad$ $\begin{array}{c} \mathrm{Tr}(A) = 8 \\ \det(A) = 17 \\ \Delta = -4 \end{array}$

SOLUTION: Using the trace, determinant and discriminant, we see that we have a spiral source, and so we expect complex conjugate eigenvalues with positive real part:

$$\lambda^2 - 8\lambda + 17 = 0 \quad \Rightarrow \quad (\lambda - 4)^2 + 1 = 0 \quad \Rightarrow \quad \lambda = 4 \pm i$$

For $\lambda = 4+i$, the second equation might be a bit easier to use: $v_1 + (3 - (4+i))v_2 = 0$, or $v_1 = (1+i)v_2$. Writing this in vector form, $\mathbf{v} = [1+i, 1]^T$ (an alternative would be $[2, 1-i]^T$)

Now we need to compute $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{4t}(\cos(t) + i\sin(t)) \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \mathrm{e}^{4t} \begin{bmatrix} \cos(t) - \sin(t) + i(\sin(t) - \cos(t)) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

In summary, the solution is:

$$\mathbf{x}(t) = \mathrm{e}^{4t} \left( C_1 \begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) + \cos(t) \\ \sin(t) \end{bmatrix} \right)$$

2. Solve the following second order IVP three ways: (i) Using methods of Chapter 3, (ii) Using the Laplace transform, and (iii) by converting it into a system of first order.

$$y'' - 2y' - 3y = 0, \quad y(0) = 1, y'(0) = 1$$

(a) Using Chapter 3, the ansatz is $y = \mathrm{e}^{rt}$, the characteristic equation is $r^2 - 2r - 3 = 0$, from which we get $r = -1, 3$. The solution is

$$y(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{3t}$$

Using the initial conditions,

$$\begin{aligned} C_1 + C_2 &= 1 \\ -C_1 + 3C_2 &= 1 \end{aligned} \quad \Rightarrow \quad C_1 = C_2 = \frac{1}{2}$$

Therefore, the specific solution to the IVP is given by

$$\frac{1}{2}e^{-t} + \frac{1}{2}e^{3t}$$

(b) Using the Laplace transform:

$$(s^2Y - s - 1) - 2(sY - 1) - 3 = 0 \quad \Rightarrow \quad (s^2 - 2s - 3)Y = s - 3 \quad \Rightarrow \quad Y = \frac{s-3}{s^2 - 2s - 3}$$

Using partial fractions, we have:

$$Y = \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-3} \quad \Rightarrow \quad y(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t}$$

(c) Using eigenvalues and eigenvectors, we let $x_1 = y$, $x_2 = y'$. Then the sytem of first order is given below. Notice that the initial values mean that $\mathbf{x}(0) = [1, 1]^T$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The trace is 2, determinant is $-3$ and the discriminant is positive, so we expect a saddle (eigenvalues with mixed signs). The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$ (the same as before), so $\lambda = -1, 3$.
For $\lambda_1 = -1$, the eigenvector is found by solving $v_1 + v_2 = 0$, or $\mathbf{v} = [1, -1]^T$.
For $\lambda_2 = 3$, the eigenvector is found by solving $-3v_1 + v_2 = 0$, or $\mathbf{v} = [1, 3]^T$.
The general solution is:

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Now, if $\mathbf{x}(0) = [1, 1]^T$, then:

$$\begin{aligned} C_1 + C_2 &= 1 \\ -C_1 + 3C_2 &= 1 \end{aligned} \quad \Rightarrow \quad C_1 = C_2 = \frac{1}{2}$$

Therefore, the solution is:

$$\mathbf{x}(t) = \frac{1}{2}e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

3. For the following *nonlinear* systems, find the equilibrium solutions, then find the general solution by looking at $dy/dx$:

$$\begin{aligned} x' &= x - xy \\ y' &= y + 2xy \end{aligned}$$

SOLUTION: The equilibrium solution(s) are found by setting the derivatives to zero:

$$\begin{aligned} x(1-y) &= 0 \\ y(1+2x) &= 0 \end{aligned}$$

Now, if $x = 0$ in the first equation, then $y = 0$ in the second, so $(0, 0)$ is one equilibrium. If $y = 1$ in the first equation, then $x = -1/2$ in the second equation, so $(-1/2, 1)$ is the second equilibrium.

Finally, taking

$$\frac{dy}{dx} = \frac{y(1+2x)}{x(1-y)} \qquad \Rightarrow \qquad \frac{1-y}{y}\,dy = \frac{1+2x}{x}\,dx$$

From which we get:

$$\ln|y| - y = \ln|x| + 2x + C$$

4. For each system $\mathbf{x}' = A\mathbf{x}$, the matrix $A$ will depend upon the parameter $\alpha$: (i) Determine the eigenvalues in terms of $\alpha$, (ii) Find the critical values of $\alpha$ where the behavior of the solution to the system changes significantly. We'll go through one or two in class.

   (a) $\begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$

   SOLUTION: Here, the trace is zero, the determinant is $5 - 4\alpha$, and the discriminant is $-4(5 - 4\alpha)$ (so it will be opposite in sign to the determinant.

   Because the trace is zero, we're on the "det$(A)$" axis in the Poincare Diagram. If the determinant is positive ($\alpha > 4/5$), then the discriminant is negative, and the origin is a center. If the determinant is negative ($\alpha < 4/5$), we have a saddle, and if $\alpha = 4/5$, we have "uniform motion".

   The eigenvalues are $\lambda = \pm\sqrt{5 - 4\alpha}$.

   (b) $\begin{bmatrix} 0 & \alpha \\ 1 & -2 \end{bmatrix}$

   We do a similar analysis here. The trace is $-2$, the determinant is $-\alpha$, and the discriminant is $4 + 4\alpha$.

   Looking on the Poincare Diagram, if the determinant is negative ($\alpha > 0$), then the origin is a saddle. If $\alpha = 0$, we have a line of stable fixed points, and if $-1 < \alpha < 0$, then the determinant is now positive, and the discriminant is negative (a spiral sink). If $\alpha = -1$, we have a degenerate sink, and if $\alpha < -1$, we have a sink.

   The eigenvalues are $\lambda = -1 \pm \sqrt{1 + \alpha}$.