

Review Questions: Exam 3

1. What is the ansatz we use for y in: Chapter 6? Section 5.2?
2. Finish the definitions:

- The Heaviside function, $u_c(t)$:
- The Dirac δ -function: $\delta(t - c)$ (Note: the Dirac function should be defined as a certain limit)
- Define the convolution: $(f * g)(t)$
- A function is of **exponential order** if:

3. Use the *definition* of the Laplace transform to determine $\mathcal{L}(f)$:

(a)

$$f(t) = \begin{cases} 3, & 0 \leq t < 2 \\ 6 - t, & t \geq 2 \end{cases}$$

(b)

$$f(t) = \begin{cases} e^{-t}, & 0 \leq t < 5 \\ -1, & t \geq 5 \end{cases}$$

4. Check your answers to Problem 3 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.
5. Write the following functions in piecewise form (thus removing the Heaviside function):

(a) $(t + 2)u_2(t) + \sin(t)u_3(t) - (t + 2)u_4(t)$ (b) $\sum_{n=1}^4 u_{n\pi}(t) \sin(t - n\pi)$

6. Determine the Laplace transform, using the table:

(a) $t^2 e^{-9t}$ (d) $e^{3t} \sin(4t)$
(b) $e^{2t} - t^3 - \sin(5t)$ (e) $e^t \delta(t - 3)$
(c) $t^2 y'(t)$ (in terms of $Y(s)$) (f) $t^2 u_4(t)$

7. Find the inverse Laplace transform, using the table:

(a) $\frac{2s - 1}{s^2 - 4s + 6}$ (d) $\frac{3s - 1}{2s^2 - 8s + 14}$
(b) $\frac{7}{(s + 3)^3}$ (e) $(e^{-2s} - e^{-3s}) \frac{1}{s^2 + s - 6}$
(c) $\frac{e^{-2s}(4s + 2)}{(s - 1)(s + 2)}$

8. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

(a) $y'' + 2y' + 2y = t^2 + 4t, y(0) = 0, y'(0) = -1$

- (b) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$
 (c) $y'' - 4y' + 4y = t^2e^t$, $y(0) = 0$, $y'(0) = 0$

9. Solve the given initial value problems using Laplace transforms:

- (a) $2y'' + y' + 2y = \delta(t - 5)$, zero initial conditions.
 (b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$
 (c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$
 (d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$
 (e) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

10. For the following, use Laplace transforms to solve, and leave your answer in the form of a convolution:

- (a) $4y'' + 4y' + 17y = g(t)$ $y(0) = 0, y'(0) = 0$
 (b) $y'' + y' + \frac{5}{4}y = 1 - u_{\pi}(t)$, with $y(0) = 1$ and $y'(0) = -1$.

11. Short Answer:

(a) $\int_0^{\infty} \sin(3t)\delta(t - \frac{\pi}{2}) dt = \underline{\hspace{4cm}}$

(b) Use Laplace transforms to solve the first order DE, thus finding which function has the Dirac function as its derivative:

$$y'(t) = \delta(t - c), \quad y(0) = 0$$

(c) Use Laplace transforms to solve for $F(s)$, if

$$f(t) + 2 \int_0^t \cos(t-x)f(x) dx = e^{-t}$$

(So only solve for the transform of $f(t)$, don't invert it back).

- (d) In order for the Laplace transform of f to exist, f must be _____
 (e) Can we assume that the solution to: $y'' + p(x)y' + q(x)y = u_3(x)$ is a power series?
 (f) Use the table to find the Laplace transform of $e^{-2t} \sinh(\sqrt{3}t)$. (Note: You don't need the definition the hyperbolic sine to answer this question).

12. Let $f(t) = t$ and $g(t) = u_2(t)$.

- (a) Use the Laplace transform to compute $f * g$.
 (b) Verify your answer by computing $f * g$ using the definition of the convolution.

13. If $a_0 = 1$, determine the coefficients a_n so that

$$\sum_{n=1}^{\infty} na_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Try to identify the series represented by $\sum_{n=0}^{\infty} a_n x^n$.

14. Write the following as a single sum in the form $\sum_{k=2}^{\infty} c_k(x-1)^k$ (with perhaps a few terms in the front):

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + x(x-2) \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

15. Characterize ALL (continuous or not) solutions to

$$y'' + 4y = u_1(t), \quad y(0) = 1, y'(0) = 1$$

16. Use the table to find an expression for $\mathcal{L}(ty')$. Use this to convert the following DE into a linear first order DE in $Y(s)$ (do not solve):

$$y'' + 3ty' - 6y = 1, y(0) = 0, y'(0) = 0$$

17. Find the recurrence relation between the coefficients for the power series solutions to the following:

(a) $2y'' + xy' + 3y = 0, x_0 = 0.$

(b) $(1-x)y'' + xy' - y = 0, x_0 = 0$

(c) $y'' - xy' - y = 0, x_0 = 1$

18. Exercises with the table:

(a) Use table entries 5 and 14 to prove the formula for 9.

(b) Show that you can use table entry 19 to find the Laplace transform of $t^2\delta(t-3)$ (verify your answer using a property of the δ function).

(c) Prove (using the definition of \mathcal{L}) table entries 12 and 13.

(d) Prove (using the definition of \mathcal{L}) a formula (similar to 18) for $\mathcal{L}(y'''(t))$.

19. Find the first 5 terms of the power series solution to $e^x y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = -1$.

20. Find the radius of convergence for all of the following, and find the interval of convergence for (b) and (d):

(a) $\sum_{n=1}^{\infty} \sqrt{n}x^n$

(c) $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$ (A little tricky)

(b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}(x+3)^n$

(d) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n5^n}$