Homework for 7.1 (Solutions)

1. Convert to a system of first order DEs:

   (a) $y'' + 9y = 0$
   SOLUTION: Let $x_1 = y$ and $x_2 = y'$. Then $x_1' = y' = x_2$, and $x_2' = y'' = -9y = -9x_1$:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -9x_1 \end{aligned}$$

   (b) $y'' + 6y' + 9y = 0$
   SOLUTION: Follow the same idea, and let $x_1 = y$ and $x_2 = y'$. Substitution gives $x_2' = y'' = -9y - 6y' = -9x_1 - 6x_2$, so:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -9x_1 - 6x_2 \end{aligned}$$

   (c) $y'' + y' - 2y = 0$

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= 2x_1 - x_2 \end{aligned}$$

2. For the previous problems, solve the systems by using methods from Chapter 3 on the second order equation.

   (a) The characteristic equation is $r^2 + 9 = 0$, or $r = \pm 3i$. The solution is therefore

   $$y = C_1 \cos(3t) + C_2 \sin(3t) \quad \Rightarrow \quad x_1 = C_1 \cos(3t) + C_2 \sin(3t)$$

   and $x_2 = y' = 3C_2 \cos(3t) - 3C_1 \sin(3t)$.

   (b) The characteristic equation is $r^2 + 6r + 9 = 0$, or $r = -3, -3$. Therefore,

   $$y = \mathrm{e}^{-3t}(C_1 + C_2 t) \quad \Rightarrow \quad x_1 = \mathrm{e}^{-3t}(C_1 + C_2 t)$$

   Therefore, $x_2 = y' = -\mathrm{e}^{-3t}(3C_1 - C_2 + 3C_2 t)$

   (c) The characteristic equation has roots $r = 1, -2$, so

   $$y = C_1 \mathrm{e}^{t} + C_2 \mathrm{e}^{-2t} \quad \Rightarrow \quad x_1 = C_1 \mathrm{e}^{t} + C_2 \mathrm{e}^{-2t}$$

   Therefore, $x_2 = y' = C_1 \mathrm{e}^{t} - 2C_2 \mathrm{e}^{-2t}$.

3. Convert the following systems to second order equations:

   (a) $$\begin{aligned} x_1' &= 2x_1 + x_2 \\ x_2' &= x_1 + x_2 \end{aligned}$$

   SOLUTION: From the first equation, $x_2 = x_1' - 2x_1$. Substitute into the second:

   $$(x_1' - 2x_1)' = x_1 + (x_1' - 2x_1) \quad \Rightarrow \quad x_1'' - 3x_1' + x_1 = 0$$

(b) $\begin{aligned} x_1' &= x_2 \\ x_2' &= 3x_1 + x_2 \end{aligned}$

SOLUTION: From the first equation, $x_2 = x_1'$. Substitute into the second:

$$(x_1')' = 3x_1 + (x_1') \quad \Rightarrow \quad x_1'' - x_1' - 3x_1 = 0$$

(c) $\begin{aligned} x_1' &= x_1 \\ x_2' &= x_1 + x_2 \end{aligned}$

SOLUTION: In this instance, we cannot use the first equation to get something to substitute– However, we can still use the second equation. In this case,

$$x_1 = x_2' - x_2$$

and now substitute into the first equation: $x_2'' - x_2' = x_2' - x_2$, or

$$x_2'' - 2x_2' + x_2 = 0$$

Side Remark: If I were actually solving this system, I would go ahead and solve for $x_1 = C_1 e^t$, then substitute that into the second equation: $x_2' = C_1 e^t + x_2$, which is a linear first order DE.

4. Solve the system by first writing the DE as dy/dx:

(a) $\begin{aligned} x' &= x \\ y' &= x + y \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{1}{x} y \quad \Rightarrow \quad \frac{dy}{dx} - \frac{1}{x} y = 1$

This is linear first order, so get the integrating factor and proceed as usual:

$$\left(\frac{y}{x}\right)' = \frac{1}{x} \quad \Rightarrow \quad \frac{y}{x} = \ln|x| + C \quad \Rightarrow \quad y = x\ln|x| + Cx$$

To solve the system directly (See the side remark above), we see that $x(t) = C_1 e^t$, so that

$$y' = C_1 e^t + y \quad \Rightarrow \quad y' = y = C_1 e^t$$

Solving, we should get $y(t) = e^t(C_1 t + C_2)$

(b) $\begin{aligned} x' &= x + y \\ y' &= x - y \end{aligned}$

(For this problem, is the DE homogeneous (the Chapter 2 version)? If so, just write how you would proceed to solve it).

SOLUTION:

$$\frac{dy}{dx} = \frac{x-y}{x+y} = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}$$

Let $v = y/x$, or $y = vx$. Then $dy/dx = v + xv'$, and:

$$v + xv' = \frac{1-v}{1+v} \quad \Rightarrow \quad xv' = \frac{1-v}{1+v} - v$$

This is a DE which is separable.