

Solutions to the Review Questions

Short Answer/True or False

1. True or False, and explain:

- (a) If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.

False: This is an isocline associated with a slope of zero, and furthermore, $y = -2t$ is not a solution, and it is not a constant.

- (b) Let $\frac{dy}{dt} = 1 + y^2$. The Existence and Uniqueness theorem tells us that the solution (for any initial value) will be valid for all t . (If true, say why. If False, solve the DE).

False. The E & U Theorem tells that a unique solution will exist for any initial condition (since $1 + y^2$ and $2y$ are continuous everywhere), but it does not say on what interval the solution will exist. For example, if we take $y(0) = y_0$ and solve, we get:

$$\int \frac{dy}{1+y^2} = \int dt \Rightarrow \tan^{-1}(y) = t + C \Rightarrow C = \tan^{-1}(y_0)$$

Therefore,

$$y = \tan(t + \tan^{-1}(y_0))$$

where

$$-\frac{\pi}{2} < t + \tan^{-1}(y_0) < \frac{\pi}{2}$$

(so that the tangent function is invertible).

- (c) If $y' = \cos(y)$, then the solutions are periodic.

FALSE. A function y is periodic if it is periodic in t , and once a function increases (for example), it cannot decrease again (since the slopes along any horizontal line are constant).

- (d) All autonomous equations are separable.

True. Any autonomous equation can be written as $y' = f(y) \cdot 1$, which is separable and $\int \frac{dy}{f(y)} = \int dt$.

- (e) All linear (first order) equations are Bernoulli.

True- It is possible to categorize them this way:

$$y' + p(t)y = g(t)y^0$$

but we typically will not write them this way (so you could argue "false" as well).

- (f) All separable equations are exact.

True. If the equation is separable, then $y' = f(y)g(x)$, which can be written:

$$\frac{1}{f(y)} \frac{dy}{dx} = g(x) \Rightarrow -g(x) + \frac{1}{f(y)} \frac{dy}{dx} = 0$$

Now, if $M(x, y) = -g(x)$, then $M_y = 0$, and $N(x, y) = 1/f(y)$ means $N_x = 0$.

2. The Existence and Uniqueness Theorems:

- Linear: $y' + p(t)y = g(t)$ at (t_0, y_0) :

If p, g are continuous on an interval I that contains t_0 , then there exists a unique solution to the initial value problem and that solution is valid for all t in the interval I .

- General Case: $y' = f(t, y)$, (t_0, y_0) :

Let the functions f and f_y be continuous in some open rectangle R containing the point (t_0, y_0) . Then there exists an interval about t_0 , $(t_0 - h, t_0 + h)$ contained in R for which a unique solution to the IVP exists.

Side Remark 1: To determine such a time interval, we must solve the DE.

Side Remark 2: We broke out the theorem in class into two components (existence and uniqueness). You can use either the theorem there or as it stated above.

3. To solve $y' = y^{1/3}$, we separate variables:

$$y^{-1/3} dy = dt$$

Before going further, it is good practice to note that the previous step is valid, *as long as* $y \neq 0$. The case that $y = 0$ can be taken separately- In fact, we see that $y(t) = 0$ is an equilibrium solution that satisfies the initial condition.

Going on, we integrate:

$$\frac{3}{2}y^{2/3} = t + C_1 \Rightarrow y^{2/3} = \frac{2}{3}t + C_2 \Rightarrow y = \left(\frac{2}{3}t + C_2\right)^{3/2}$$

We can solve for C_2 using the initial condition: $0 = C_2$, so that

$$y = \left(\frac{2t}{3}\right)^{3/2}$$

We can verify that this is indeed a solution by substituting it back into the DE (not necessary; just a way of double-checking yourself):

$$y' = \frac{3}{2} \left(\frac{2t}{3}\right)^{1/2} \cdot \frac{2}{3} = \left(\frac{2t}{3}\right)^{1/2}$$

And on the other hand,

$$y^{1/3} = \left[\left(\frac{2t}{3}\right)^{3/2}\right]^{1/3} = \left(\frac{2t}{3}\right)^{1/2}$$

Therefore, this is indeed a second solution to the IVP.

Of course, the Existence and Uniqueness Theorem cannot be “violated” since it is a theorem, but in this case, the *hypotheses* are not satisfied:

$$y' = f(t, y) \Rightarrow f(t, y) = y^{1/3}$$

In this case, f is continuous at $(0, 0)$ but $\partial f/\partial y$ is not.

Solve:

1. $\frac{dy}{dx} = \frac{x^2 - 2y}{x}$

Linear: $y' + \frac{2}{x}y = x$ Solve with an integrating factor of x^2 to get:

$$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

2. $(x + y) dx - (x - y) dy = 0$. Hint: Let $v = y/x$.

Given the hint, rewrite the DE:

$$\frac{dy}{dx} = \frac{x + y}{x - y} = \frac{1 + (y/x)}{1 - (y/x)} = \frac{1 + v}{1 - v}$$

With the substitution $xv = y$, we get the substitution for dy/dx :

$$v + xv' = y'$$

So that the DE becomes:

$$v + xv' = \frac{1 + v}{1 - v} \Rightarrow xv' = \frac{1 + v}{1 - v} - v = \frac{1 + v - v(1 - v)}{1 - v} = \frac{1 + v^2}{1 - v}$$

The equation is now separable:

$$\frac{1-v}{1+v^2} dv = \frac{1}{x} dx \Rightarrow \int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv = \ln|x| + C$$

Therefore,

$$\tan^{-1}(v) - \frac{1}{2} \ln(1+v^2) = \ln|x| + C$$

Lastly, back-substitute $v = y/x$.

3. $\frac{dy}{dx} = \frac{2x+y}{3+3y^2-x} \quad y(0) = 0.$

This is exact. The solution is, with $y(0) = 0$,

$$-x^2 - xy + 3y + y^3 = 0$$

4. $\frac{dy}{dx} = -\frac{2xy + y^2 + 1}{x^2 + 2xy}$

This is exact. The solution is: $x^2y + xy^2 + x = c$

5. $\frac{dy}{dt} = 2 \cos(3t) \quad y(0) = 2$

This is linear and separable. $y(t) = \frac{2}{3} \sin(3t) + 2$, and the solution is valid for all time.

6. $y' - \frac{1}{2}y = 0 \quad y(0) = 200$. State the interval on which the solution is valid.

This is linear and separable. As a linear equation, the solution will be valid on all t (since $p(t) = -\frac{1}{2}$).

The solution is $y(t) = 200e^{(1/2)t}$

7. This is separable (or Bernoulli):

$$\int y^{-2} dy = \int (1-2x) dx \Rightarrow -\frac{1}{y} = x - x^2 + C$$

Put in the initial condition (IC): $6 = 0 + C$. Now finish solving explicitly:

$$y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)}$$

The solution is valid on the interval $(-2, 3)$.

8. $y' - \frac{1}{2}y = e^{2t} \quad y(0) = 1$

This is linear (but not separable). $y(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{(1/2)t}$

9. $y' = \frac{1}{2}y(3-y)$

Autonomous (and separable). Integrate using partial fractions:

$$\int \frac{1}{y(3-y)} dy = \frac{1}{2} \int dt$$

Simplify your answer for y by dividing numerator and denominator appropriately to get:

$$y(t) = \frac{3}{(1/A)e^{-(3/2)t} + 1}$$

10. $\sin(2t) dt + \cos(3y) dy = 0$

Separable (and/or exact): $-\frac{1}{2} \cos(2t) + \frac{1}{3} \sin(3y) = C$

11. $y' = xy^2$

Separable: $y = \frac{1}{-(1/2)x^2 - C}$

12. $\frac{dy}{dx} - \frac{3}{2x}y = \frac{2x}{y}$ (Hint: Think Bernoulli)

SOLUTION: Multiply by y to get

$$yy' - \frac{3}{2x}y^2 = 2x$$

So (following what we did for the general Bernoulli eqn), let $v = y^2$, and therefore $v' = 2yy'$. Multiply by 2 to get the right form, then substitute

$$2yy' - \frac{3}{x}y^2 = 4x \quad \Rightarrow \quad v' - \frac{3}{x}v = 4x$$

Now it is a standard linear DE. Solving, we get $v = -4x^3 + Cx^3$, and

$$y^2 = -4x^3 + Cx^3$$

(We'll leave in implicit form).

13. $yy'' = (y')^2$ With the hint, we need to find a substitution for y'' (and then leave the equation as a DE with unknown function p with variable y):

$$\frac{dy}{dt} = p(y) \quad \Rightarrow \quad \frac{d^2y}{dt^2} = \frac{dp}{dy} \cdot \frac{dy}{dt} = \frac{dp}{dy}p(y)$$

Now, substitute in the expressions and divide by $yp(y)$:

$$y \frac{dp}{dy}p(y) = (p(y))^2 \quad \Rightarrow \quad \frac{dp}{dy} = \frac{1}{y}p(y) \quad \Rightarrow \quad \int \frac{1}{p} dp = \int \frac{1}{y} dy \quad \Rightarrow \quad \ln |p| = \ln |y| + C \quad \Rightarrow \quad p = Ay$$

where A is a constant of integration. Now, substitute again with $p = y'$, and we get:

$$y' = Ay \quad \Rightarrow \quad y = Pe^{At}$$

where P is another constant of integration.

We can verify our answer as well: $y' = APe^{At}$, and $y'' = A^2Pe^{At}$ so that

$$yy'' = Pe^{At} \cdot A^2Pe^{At} = A^2P^2e^{2At}$$

and this expression is also $(y')^2$.

14. $y' + 2y = g(t)$ with $y(0) = 0$ and $g(t) = 1$ on $0 \leq t \leq 1$ and zero elsewhere.

SOLUTION: This is similar to Exercise 33, Section 2.4. In this case, we go ahead and solve starting at time 0:

$$y' + 2y = 1 \quad \Rightarrow \quad y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

and this is valid for $0 \leq t \leq 1$. When we hit $t = 1$, the dynamics change to:

$$y' + 2y = 0 \quad \Rightarrow \quad y(t) = Pe^{-2t}$$

Now we will typically choose the constants so that y is continuous. Therefore, using our previous function, $y(1) = (1 - e^{-2})/2$, and our current function: $y(1) = Pe^{-2}$, or

$$P = \frac{e^2 - 1}{2}$$

Therefore, the overall solution to the DE would be:

$$y(t) = \begin{cases} (1 - e^{-2t})/2 & \text{if } 0 \leq t \leq 1 \\ ((e^2 - 1)/2)e^{-2t} & \text{if } 1 > t \end{cases}$$

Just for fun, the direction field and solution curve are plotted in Figure ??.

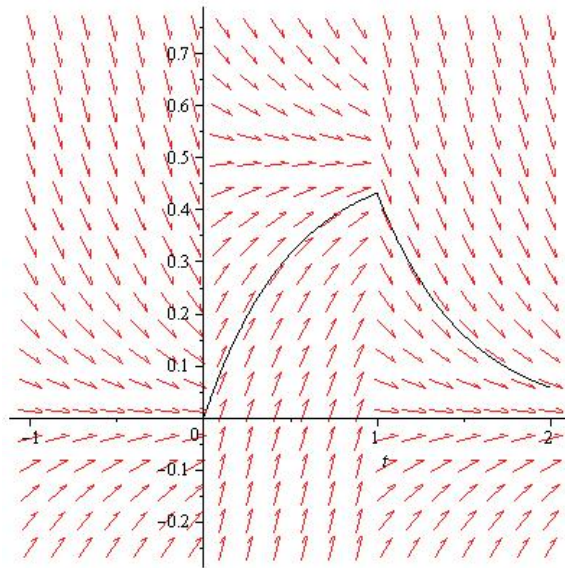


Figure 1: Direction field and solution curve for Exercise 14. Note how the solution approaches one equilibrium until $g(t)$ changes, then it goes to the new equilibrium.

Misc.

1. Construct a linear first order differential equation whose general solution is given by:

(a) $y(t) = t - 3 + \frac{C}{t^2}$

SOLUTION: Construct y' . The idea will be to produce a linear DE. Therefore, we need to construct y' and compare it to y :

$$y' = 1 - 2Ct^{-3}$$

Add this to some multiple (t 's are allowed) of y to get rid of the arbitrary constant. In this case,

$$y' + \frac{2}{t}y = (1 - 2Ct^{-3}) + 2 - \frac{6}{t} + 2Ct^{-3} = 3 - \frac{6}{t}$$

or,

$$ty' + 2y = 3t - 6$$

(b) $y(t) = 2 \sin(3t) + Ce^{-2t}$

SOLUTION: Same idea as before. Try to get y and y' together in such a way as to cancel out the arbitrary C :

$$y' = 6 \cos(3t) - 2Ce^{-2t}$$

so that: $y' + 2y = 4 \sin(3t) + 6 \cos(3t)$.

2. Construct a linear first order DE so that all solutions tend to $y = 3$ as $t \rightarrow \infty$.

SOLUTION: One way to think about it is to construct $y' = ay + b$, which is also autonomous- In fact, we are saying that we need a line in the plane, through $y = 3$ so that the equilibrium is stable- That is any line through $(0, 3)$ with a negative slope. For example, $y' = -y + 3$

3. Suppose we want to construct a population model so that there is a logistic population growth-

- (a) That is, there is an environmental carrying capacity of 100. Construct an appropriate (autonomous) model.

SOLUTION: In the (y, y') plane, we are looking at an upside down parabola that goes through $y = 0$ and $y = 100$. One model is therefore

$$y' = y(100 - y)$$

- (b) Using your model, now assume there is a continuous “harvest” (of k units per time period). How does that effect the model- In particular, is there a critical value of k over which the population will be extinct? If so, find it.

SOLUTION: This subtracts k from our population:

$$y' = y(100 - y) - k$$

We see that for some value of k , the vertex of the upside parabola is exactly on the y -axis. We can solve this by completing the square- That is, we should be able to write y' as a perfect square:

$$-y^2 + 100y + k \rightarrow -(y^2 - 100y + 50^2) = -(y - 50)^2$$

Therefore, $k = 50^2 = 2500$. This is the maximum allowable harvest before the ecosystem collapses.

4. Suppose we have a tank that contains M gallons of water, in which there is Q_0 pounds of salt. Liquid is pouring into the tank at a concentration of r pounds per gallon, and at a rate of γ gallons per minute. The well mixed solution leaves the tank at a rate of γ gallons per minute.

Write the initial value problem that describes the amount of salt in the tank at time t , and solve:

$$\frac{dQ}{dt} = r\gamma - \frac{\gamma}{M}Q, \quad Q(0) = Q_0$$

The solution is:

$$Q = rM + (Q_0 - rM)e^{-(\gamma/M)t}$$

5. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on Q_0 ? Does this make sense?

Note that the differential equation for Q is autonomous, so we could do a phase plot (line with a negative slope). Or, we can just take the limit as $t \rightarrow \infty$ and see that $Q \rightarrow rM$. This does not necessarily depend on Q_0 ; if Q_0 starts at equilibrium, rM , then Q is constant.

It does make sense. The incoming concentration of salt is r pounds per gallon, so we would expect the long term concentration to be the same, $rM/M = r$.

6. Modify problem 5 if: $M = 100$ gallons, $r = 2$ and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if $Q_0 = 50$.

$$\frac{dQ}{dt} = 4 - \frac{3}{100-t}Q \quad Q(0) = 50$$

This goes from being autonomous to linear. In this case, use an integrating factor,

$$e^{\int p(t) dt} = e^3 \int \frac{1}{100-t} dt = e^{-3 \ln |100-t|} = (100+t)^{-3} \quad t > 100$$

Going back to the DE

$$\left(\frac{Q}{(100-t)^3} \right) = 4(100-t)^{-3} \Rightarrow Q = 2(100-t) + C(100-t)^3$$

Continuing, we get:

$$Q(t) = 2(100-t) - \frac{150}{100^3}(100-t)^3$$

7. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v$, find the initial value problem (and solve it) for the velocity at time t .

The general model is: $mv' = mg - kv$. In this case, $m = 1$, $g = 9.8$ and $k = 1/2$. Therefore,

$$v' = 9.8 - \frac{1}{2}v$$

Which is linear (and autonomous). Since the object is being dropped, the initial velocity is zero.

Solve it:

$$v(t) = 19.6 \left(1 - e^{-(1/2)t} \right)$$

8. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of k dollars per month, set up a model for how much you owe at time t in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

SOLUTION: First, notice that the units of time are mixed- The interest rate is an annual rate, but k is in dollars per month. We should first decide on what the units of time should be. The answer below will assume that we are working with time in *years*, so that the annual payments are $12k$ dollars per year (in a continuous fashion).

Therefore, the model is $S' = rS - 12k$, where S will be the amount owing, r is the annual interest rate and k is the rate for the continuous payment (per month). Then using $S(0) = S_0$, we can write the solution as

$$S(t) = \frac{12k}{r} + \left(S_0 - \frac{12k}{r} \right) e^{rt}$$

Substituting in the values for r and S_0 , and $t = 10$, we can solve the equation for k :

$$0 = 240k + (10000 - 240k)e^{\frac{1}{2}}$$

Extra: If we go ahead and solve, we get a monthly payment rate of $k \approx \$105.90$.

9. Show that the IVP $xy' = y - 1$, $y(0) = 2$ has no solution. (Note: Part of the question is to think about how to show that the IVP has no solution).

SOLUTION: We'll try to solve it, and see what happens. The DE is separable. Notice that the function $f(x, y)$ from the E& U theorem is $(y - 1)/x$, which is not continuous at $x = 0$...

Continuing as usual:

$$\int \frac{1}{y-1} dy = \int \frac{1}{x} dx \quad \Rightarrow \quad \ln|y-1| = \ln|x| + C$$

Exponentiate both sides to get

$$y - 1 = Ax \quad \Rightarrow \quad y = Ax + 1$$

There is no choice of A that will satisfy the initial condition,

$$2 = A \cdot 0 + 1$$

10. Suppose that a certain population grows at a rate proportional to the square root of the population. Assume that the population is initially 400 (which is 20^2), and that one year later, the population is 625 (which is 25^2). Determine the time in which the population reaches 10000 (which is 100^2)

SOLUTION: If $P(t)$ is the population at time t , then the first part of the statement translates to:

$$\frac{dP}{dt} = kP^{1/2}$$

where k is the constant of proportionality. This is separable (and autonomous), so:

$$\int P^{-1/2} dP = \int k dt \quad \Rightarrow \quad P^{1/2} = (k/2)t + C$$

Given the initial population, we can solve for C , given the second piece of info, we can solve for k :

$$P(0) = 20^2 \quad \Rightarrow \quad 20 = (k/2)(0) + C \quad \Rightarrow \quad C = 20$$

and $P(1) = 25^2$ gives:

$$25 = (k/2) + 20 \quad \Rightarrow \quad k = 10$$

Therefore, our model is:

$$P(t) = (5t + 20)^2$$

Solving for the last part, $P(t) = 100^2$, we have

$$100 = 5t + 20 \quad \Rightarrow \quad t = 16$$

11. Consider the sketch below of $F(y)$, and the differential equation $y' = F(y)$.

(a) Find and classify the equilibrium.

SOLUTION: From the sketch given, $y = 0$ is asymptotically stable and $y = 1$ is semistable.

(b) Find intervals (in y) on which $y(t)$ is concave up.

SOLUTION: Examine the intervals $y < 0$, $0 < y < 1/3$, $1/3 < y < 1$ and $y > 1$ separately. The function y will be concave up when dF/dy and F both have the same sign. This happens when F is either increasing and positive (which happens nowhere) or decreasing and negative:

$$0 < y < \frac{1}{3} \quad y > 1$$

(c) Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. See the figure below.

(d) Find an appropriate polynomial for $F(y)$.

SOLUTION: One example is

$$y' = -y(y - 1)^2$$

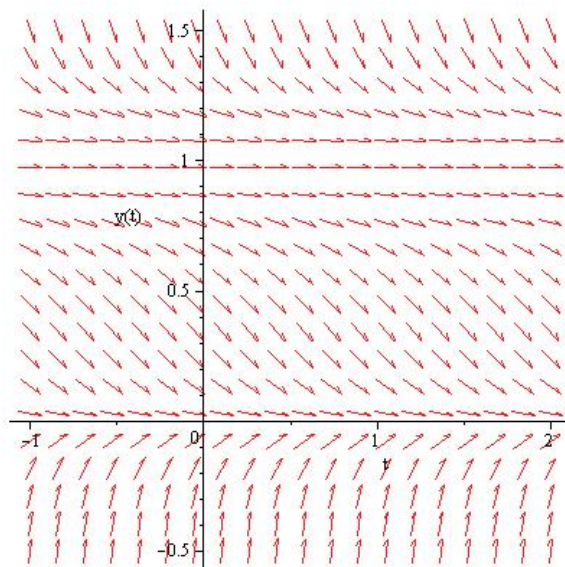


Figure 2: Figure for Exercise 9.

12. Consider the DE: $y dx + (2x - ye^y) dy = 0$. Show that the equation is not exact, but becomes exact if you assume there is an integrating factor in terms of y alone (Hint: Find the integrating factor first).

SOLUTION: Let μ be the integrating factor, and assume μ is a function of y alone. Then

$$(\mu M)_y = \mu' \cdot M + \mu \cdot M_y = y\mu' + \mu$$

And

$$(\mu N)_x = 0 \cdot N + \mu \cdot 2$$

Setting these equal, we have the DE:

$$y\mu' = \mu \quad \Rightarrow \quad \int \frac{1}{\mu} d\mu = \int \frac{dy}{y}$$

or $\mu = y$. We can verify our answer, since $y^2 dx + (2xy - y^2 e^y) dy = 0$ should now be exact.

13. Given the direction field below, find a differential equation that is consistent with it.

SOLUTION: Draw the corresponding figure in the (y, y') plane first. There we see that $y = 0$ is unstable (make it a linear crossing), and $y = 2$ is stable, $y = 4$ is unstable. From the figure,

$$y' = y(y - 2)(y - 4)$$

will work.

14. Consider the direction field below, and answer the following questions:

(a) Is the DE possibly of the form $y' = f(t)$?

SOLUTION: No. The isoclines would be vertical (consider, for example, a vertical line at $t = -3$; the slopes are clearly not equal).

(b) Is the DE possible of the form $y' = f(y)$?

SOLUTION: No. The isoclines would be horizontal (for example, look at a horizontal line at $y = 1$ - Some slopes are zero, others are not).

(c) Is there an equilibrium solution? (If so, state it):

SOLUTION: Yes- At $y = 0$.

(d) Draw the solution corresponding to $y(-1) = 1$.

SOLUTION: Just draw a curve consistent with the arrows shown.

15. Evaluate the following integrals:

SOLUTIONS: You should be able to integrate by parts and use partial fractions fairly quickly at this point. Try checking your answers on Wolfram Alpha (for example, `integrate x^3exp(2x)`)

Some notes: The first integral should be done using a table, and the last should simplify a lot.

$$\int x^3 e^{2x} dx \quad \int \frac{x}{(x-1)(2-x)} dx \quad e^{-3} \int dt/t$$