Solutions to the Homework
Replaces Section 3.7, 3.8

1. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$

   SOLUTION: With no damping, $mu'' + ku = 0$ has solution

$$u(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

   so the period is given below. We also note that $mg - kL = 0$, and this equation yields the desired substitution:

$$P = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{and} \qquad mg = kL \quad \Rightarrow \quad \frac{k}{m} = \frac{g}{L}$$

2. Convert the following to $R\cos(\omega t - delta)$

   SOLUTION: Given $A\cos(\omega t) + B\sin(\omega t)$, then $R = \sqrt{A^2 + B^2}$ and $\delta = \tan^{-1}(B/A)$. For δ, use a "four quadrant inverse" by adding or subtracting π when necessary- when (A, B) as a point in the plane is in quadrant II or III.

   I included these conversions because the formulas we use are identical to those we used in converting $a + ib$ in the complex plane to polar form, $re^{i\theta}$.

   (a) $u = 2\cos(3t) + \sin(3t)$
   SOLUTION: $R = \sqrt{5}$ and $\omega = 3$. The angle δ is computed as the argument of the point $(2, 1)$, which you can leave as $\delta = \tan^{-1}(1/2)$:

$$2\cos(3t) + \sin(3t) = \sqrt{5}\cos(3t - \tan^{-1}(1/2))$$

   (b) $u = -2\pi\cos(\pi t) - \pi\sin(\pi t)$
   SOLUTION: Same idea, but note that $(-2\pi, -\pi)$ is a point in Quadrant III, so we add (or subtract) π:

$$R = \pi\sqrt{5} \qquad \text{and} \qquad \delta = \tan^{-1}(1/2) + \pi \qquad \text{and} \qquad \omega = \pi$$

   (c) $u = 5\sin(t/2) - \cos(t/2)$
   SOLUTION: Did you notice I reversed the sine and cosine on you (that was a mistake, but maybe it was a helpful one). The value of R and $omega$ would be the same either way, but δ changes:

$$R = \sqrt{26} \qquad \omega = \frac{1}{2}$$

   For δ, notice that our "point" is $(-1, 5)$ which is in Quadrant II, so add π:

$$\sqrt{26}\cos\left(\frac{t}{2} - \tan^{-1}(-5) - \pi\right)$$

3. Show that in the case of beating,

$$y'' + \omega_0^2 y = a\,\cos(\omega t)$$

the general solution is indeed of the form

$$y(t) = c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t) + \frac{a}{\omega_0^2 - \omega^2} \cos(\omega t)$$

(You don't have to answer the second part of the question).

SOLUTION: The roots to the characteristic equation are $r = \pm\omega_0 i$, so by the Method of Undetermined coefficients, if $\omega \neq \omega_0$, the particular part of the solution is:

$$\begin{array}{rl} y_p'' & = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) \\ +\omega_0^2 y_p & = \omega_0^2(A\cos(\omega t) + B\sin(\omega t)) \\ \hline a\cos(\omega t) & = A(\omega_0^2 - \omega^2)\cos(\omega t) + B(\omega_0^2 - \omega^2)\sin(\omega t) \end{array}$$

Therefore, $B = 0$ and $A = \frac{a}{\omega_0^2 - \omega^2}$, and the full solution to the given ODE is:

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{a}{\omega_0^2 - \omega^2} \cos(\omega t)$$

4. (Exercise 7, Section 3.7- Set up by hand, let Maple solve it.)

NOTE: For an exam or quiz, I would remind you that $g = 32 ft/sec^2$.

The set up: The spring constant is found by recalling that $mg - kL = 0$. Then, solving for k, we get: $k = mg/L$. In this problem, the weight of 3 pounds is mg, and the length should be in feet: $L = 1/4$, giving $k = 3/(1/4) = 12$. Using $g = 32$ ft/sec^2, the mass is $32m = 3$, of $m = 3/32$. Since there is no damping, we have:

$$\frac{3}{32}u'' + 12u = 0$$

We could continue by hand, but the numbers are messy (the book authors wanted you to use Maple to solve it).

5. (Exercise 9, Section 3.7- Set up by hand, let Maple solve it.)

Exercise 9 is similar- Careful with units (I'll keep units consistent in my questions to you as meters, kg and seconds). In this case, its probably easiest to use centimeters, grams and second. In that case, $g = 980$ cm/sec^2, and we need to solve for the spring constant:

$$m = 20 \qquad \gamma = 400 \qquad mg = kL \quad \Rightarrow \quad 20 \cdot 980 = k \cdot 5$$

Therefore,

$$20u'' + 400u' + 3920u = 0 \qquad u(0) = 2 \qquad u'(0) = 0$$

We can stop there doing this by hand (again, the authors meant for us to use Maple from here).

6. Find the general solution of the given differential equation:

$$y'' + 3y' + 2y = \cos(t)$$

First, get the homogeneous part of the solution by solving the characteristic equation:

$$r^2 + 3r + 2 = 0 \quad \Rightarrow \quad (r+2)(r+1) = 0 \quad \Rightarrow \quad r = -1, -2$$

Therefore, $y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$. You could solve for the particular solution one of two ways- Undetermined Coefficients or using Variation of Parameters. For extra practice, you might try both.

SOLUTION:

$$y_p(t) = A\cos(t) + B\sin(t) \quad y_p' = B\cos(t) - A\sin(t) \quad y_p'' = -A\cos(t) - B\sin(t)$$

so that, looking at the coefficients of $\cos(t)$ and $\sin(t)$, we have:

$$(-A + 3B + 2A)\cos(t) + (-B - 3A + 2B)\sin(t) = \cos(t)$$

Therefore, by Cramer's Rule:

$$\begin{array}{rl} A + 3B & = 1 \\ -3A + B & = 0 \end{array} \quad \Rightarrow \quad A = \frac{1}{10} \quad B = \frac{3}{10}$$

The solution is $C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$

7. What is the *transient solution*? What is the *steady state solution*? (See section 3.8)

   This will not be on the exam.

8. Pictured below are the graphs of several solutions to the differential equation:

   This will not be on the exam.

9. Recall that

$$\mathrm{Real}(e^{i\theta}) = \cos(\theta) \qquad \mathrm{Imag}(e^{i\theta}) = \sin(\theta)$$

   This will not be on the exam.

10. Fill in the question marks with the correct expression:

    Given the undamped second order differential equation, $y'' + \omega_0^2 y = A\cos(\omega t)$, we see "beating" if ($|\omega - \omega_0|$ is small) In particular, the longer period wave has a period that gets **longer** as $\omega \to \omega_0$, and its amplitude gets **bigger**

    NOTE: Beating and resonance will be on the exam, but the transition from beating to resonance (as in this question) will not be.

11. Find the solution to $y'' + 9y = 2\cos(3t)$, $y(0) = 0$, $y'(0) = 0$ by first solving the more general equation: $y'' + 9y = 2\cos(at)$, $y(0) = 0$, $y'(0) = 0$, then take the limit of your solution as $a \to 3$.

Solution: First, the homogeneous part of the solution is:

$$y_h(t) = C_1 \cos(3t) + C_2 \sin(3t)$$

The particular part is (differentiate to substitute into the DE):

$$y_p = A\cos(at) + B\sin(at) \qquad y_p'' = -Aa^2\cos(at) - Ba^2\sin(at)$$

And,

$$y_p'' + 9y_p = A\left(-a^2+9\right)\cos(at) + B\left(-a^2+9\right) = 2\cos(at) \quad \Rightarrow \quad A = \frac{2}{9-a^2} \quad B = 0$$

The particular part of the solution is:

$$y_p(t) = \frac{2}{9-a^2}\cos(at)$$

Put everything together to solve with the initial conditions, $y(0) = 0$ and $y'(0) = 0$:

$$y = C_1\cos(3t) + C_2\sin(3t) + \frac{2}{9-a^2}\cos(at) \quad \Rightarrow \quad 0 = C_1 + \frac{2}{9-a^2} \quad \Rightarrow \quad C_1 = -\frac{2}{9-a^2}$$

and, using $y'(0) = 0$,

$$0 = 0 + 3C_2 + 0 \quad \Rightarrow \quad C_2 = 0$$

Therefore, the overall solution to the IVP is:

$$y(t) = \frac{2}{9-a^2}\left(\cos(at) - \cos(3t)\right)$$

Take the limit as $a \to 3$ using L'Hospital's rule:

$$\lim_{a\to 3} \frac{2(\cos(at) - \cos(3t))}{9-a^2} = \lim_{a\to 3} \frac{-2t\sin(at)}{-2a} = \frac{-2t\sin(3t)}{(-2)(3)} = \frac{1}{3}t\sin(3t)$$

12. (Exercise 6, Section 3.8: Do this one by hand.)

Set up the constants first- Use meters, kilograms and seconds so that they all line up- In that case,

$$m = 5 \qquad k(0.1) = 5(9.8) \Rightarrow k = 490$$

For the damping, remember that we assume that damping is proportional to the velocity. Therefore, what's given in the text can be used to solve for γ. First we change 4 cm/sec to m/sec: 0.04 m/s, and:

$$0.04\gamma = 2 \quad \Rightarrow \quad \gamma = 50$$

Therefore,

$$5u'' + 50u' + 490u = 10\sin\left(\frac{t}{2}\right)$$

with $u(0) = 0$ and $u'(0) = 0.03$. That's as far as we need to go.

13. (Exercise 10, Section 3.8).

We will set it up by hand (we won't do the Maple for the exam). In this case, we'll actually solve it. The author is mixing units again, so use feet, pounds and seconds:

We see that $mg = 8$ with $g = 32$, and $mg - kL = 0$, so that we can solve for the spring constant:

$$8 - \frac{k}{2} = 0 \quad \Rightarrow \quad k = 16$$

And $m = \frac{8}{32} = \frac{1}{4}$ gives:

$$\frac{1}{4}u'' + 16u = 8\sin(8t) \quad \Rightarrow \quad u'' + 64u = 32\sin(8t)$$

Using the method of undetermined coefficients, the particular part of the solution is found by guessing, then substitute the guess into the DE:

$$u_p = t(A\cos(8t) + B\sin(8t)) \quad \Rightarrow$$

$$\begin{array}{rl} u_p'' & = (16B - 64At)\cos(8t) + (-16A - 64Bt)\sin(8t) \\ 64u_p & = 64At\cos(8t) + 64Bt\sin(t) \\ \hline 32\sin(8t) & = 16B\cos(8t) - 16A\sin(8t) \end{array}$$

From which we get $A = -2$ and $B = 0$ so that the overall solution so far is:

$$u(t) = C_1\cos(8t) + C_2\sin(8t) - 2t\cos(8t)$$

Using the initial conditions $u(0) = \frac{1}{4}$ and $u'(0) = 0$, we get:

$$\frac{1}{4} = C_1 = C_2$$

14. (Exercise 17, Section 3.8: Set up by hand.)

$$u'' + \frac{1}{4}u' + 2u = 2\cos(\omega t) \qquad u(0) = 0 \quad u'(0) = 2$$

You should be able to get the homogeneous part of the solution:

$$r^2 + \frac{1}{4}r + 2 = 0 \quad \Rightarrow \quad r^2 + \frac{1}{4}r + \frac{1}{64} = -\frac{127}{64} \quad \Rightarrow \quad r = -\frac{1}{8} \pm \frac{\sqrt{127}}{8}i$$

And state the form of the particular part of the solution:

$$y_p = A\cos(\omega t) + B\sin(\omega t)$$