

Example

Invert:

$$\frac{(s-2)e^{-s}}{s^2-4s+3} = e^{-s}H(s)$$

Symbolically, we know the inverse is: $u_1(t)h(t-1)$.

Now we compute $h(t)$:

$$H(s) = \frac{s-2}{(s-1)(s-3)} = \frac{1}{2} \frac{1}{s-1} + \frac{1}{2} \frac{1}{s-3}$$

so that the inverse is:

$$h(t) = \frac{1}{2}e^t + \frac{1}{2}e^{3t}$$

Finally, using table entry #13, the inverse is $u_1(t)h(t-1)$.

Rule of thumb for the denominator: If the polynomial is factorable, then factor it (and use partial fractions). If it is irreducible, then complete the square.

Example

Invert:

$$\frac{(e^{-5s} - e^{-20s})(s + 1)}{2s^2 + s + 2} = (e^{-5s} - e^{-20s}) H(s)$$

Then, Table Entry 13: $u_5(t)h(t - 5) - u_{20}(t)h(t - 20)$

We now invert $H(s)$:

$$\begin{aligned}\frac{s+1}{2s^2+s+2} &= \frac{1}{2} \cdot \frac{s+1}{s^2+\frac{1}{2}s+1} = \frac{1}{2} \cdot \frac{s+1}{s^2+\frac{1}{2}s+\frac{1}{16}+\frac{15}{16}} = \\ \frac{1}{2} \left[\frac{s+1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} \right] &= \frac{1}{2} \left[\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} + \frac{3}{4} \frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} \right] = \\ \frac{1}{2} \left[\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} + \frac{3}{4} \cdot \sqrt{\frac{16}{15}} \frac{\sqrt{\frac{15}{16}}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} \right]\end{aligned}$$

Finally:

$$h(t) = \frac{1}{2}e^{-t/4} \left[\cos\left(\frac{\sqrt{15}}{4}t\right) + \frac{3}{\sqrt{15}} \sin\left(\frac{\sqrt{15}}{4}t\right) \right]$$

Example

Solve, and plot the solution (technology required):

$$y'' + 4y = u_{\pi}(t) - u_{3\pi}(t) \quad y(0) = 0, y'(0) = 0$$

$$(s^2 + 4)Y = \frac{e^{-\pi s} - e^{-3\pi s}}{s} \Rightarrow Y = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2 + 4)}$$

where, through partial fractions:

$$H(s) = \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{1}{4} \cdot \frac{1}{s} - \frac{1}{4} \cdot \frac{s}{s^2 + 4}$$

so that

$$h(t) = \frac{1}{4} - \frac{1}{4} \cos(2t)$$

The solution is therefore

$$y(t) = u_{\pi}(t)h(t - \pi) - u_{3\pi}(t)h(t - 3\pi)$$

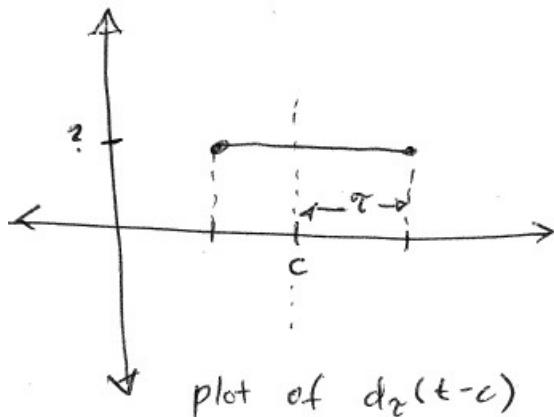
To understand this solution, we can write it piecewise (and note that $\cos(2t)$ is periodic with period π). Before summarizing it, we see that between times π and 3π , we have ($\cos(2t)$ is periodic with period π):

Intro to 6.5

We want a function that will model a finite force that occurs over a very short period of time...

Start with:

$$d_{\tau}(t - c) = \begin{cases} 1/(2\tau) & \text{if } t \in [c - \tau, c + \tau] \\ 0 & \text{otherwise} \end{cases}$$



The *impulse* is the integral of force.

By design, d_τ has a unit impulse:

$$\int_{-\infty}^{\infty} d_\tau(t - c) dt = \frac{1}{2\tau} \int_{c-\tau}^{c+\tau} 1 dt = 1$$

Definition: The Dirac Delta Function:

$$\delta(t - c) = \lim_{\tau \rightarrow 0} d_\tau(t - c)$$

Intuitively,

$$\delta(t - c) = \begin{cases} \text{"}\infty\text{"} & \text{if } t = c \\ 0 & \text{elsewhere} \end{cases}$$

However, $\delta(t - c)$ still has a *unit impulse*.

There are three main properties of $\delta(t - c)$:

- ▶ $\int_{-\infty}^{\infty} \delta(t - c) dt = 1$
- ▶ $\int_{-\infty}^{\infty} \delta(t - c)f(t) dt = f(c)$
- ▶ Lemma: $\delta(t - c)f(t) = \delta(t - c)f(c)$
- ▶ $\mathcal{L}(\delta(t - c)) = \int_0^{\infty} e^{-st} \delta(t - c) dt = e^{-cs}$

Let $y'(t) = \delta(t - c)$ with zero ICs. Solve for y using Laplace transforms.

SOLUTION:

$$sY = e^{-cs} \quad \Rightarrow \quad Y = \frac{e^{-cs}}{s} \quad \Rightarrow \quad y(t) = u_c(t)$$

Therefore,

“The ‘derivative’ of the Heaviside function is the Dirac function...”

Mass-spring with zero ICs and strike a hammer (unit impulse) at times $0, 2\pi, 4\pi$, and so on:

$$y'' + y = \delta(t) + \delta(t - 2\pi) + \delta(t - 4\pi) + \cdots = \sum_{n=0}^{\infty} \delta(t - 2n\pi)$$

SOLUTION:

$$(s^2 + 1)Y = \sum_{n=0}^{\infty} e^{-2\pi ns} \quad \Rightarrow \quad Y = \sum_{n=0}^{\infty} \frac{e^{-2\pi ns}}{s^2 + 1}$$

$$y(t) = \sum_{n=0}^{\infty} u_{2\pi n}(t) \sin(t - 2n\pi)$$

But, $\sin(t - 2n\pi) = \sin(t)$, so therefore:

$$y(t) = \sin(t) + u_{2\pi}(t) \sin(t) + u_{4\pi}(t) \sin(t) + u_{6\pi}(t) \sin(t) + \cdots$$

Writing this piecewise:

$$y(t) = \begin{cases} \sin(t) & \text{if } 0 \leq t < 2\pi \\ 2\sin(t) & \text{if } 2\pi \leq t < 4\pi \\ 3\sin(t) & \text{if } 4\pi \leq t < 6\pi \\ \vdots & \end{cases}$$