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Finally, using table entry #13, the inverse is $u_1(t)h(t-1)$.

Rule of thumb for the denominator: If the polynomial is factorable, then factor it (and use partial fractions). If it is irreducible, then complete the square.

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Then, Table Entry 13: $u_5(t)h(t-5) - u_{20}(t)h(t-20)$

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Finally:

$$h(t) = \frac{1}{2}\mathrm{e}^{-t/4}\left[\cos\left(\frac{\sqrt{15}}{4}t\right) + \frac{3}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{4}t\right)\right]$$

Example

Solve, and plot the solution (technology required):

$$y'' + 4y = u_\pi(t) - u_{3\pi}(t) \quad y(0) = 0, y'(0) = 0$$

$$(s^2+4)Y = \frac{e^{-\pi s} - e^{-3\pi s}}{s} \quad \Rightarrow \quad Y = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2+4)}$$

where, through partial fractions:

$$H(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{s}{s^2+4}$$

so that

$$h(t) = \frac{1}{4} - \frac{1}{4}\cos(2t)$$

The solution is therefore

$$y(t) = u_\pi(t)h(t-\pi) - u_{3\pi}(t)h(t-3\pi)$$

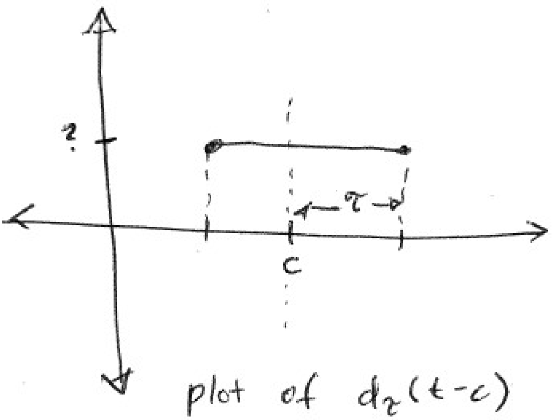
To understand this solution, we can write it piecewise (and note that $\cos(2t)$ is periodic with period π). Before summarizing it, we see that between times π and 3π, we have ($\cos(2t)$ is periodic with period π):

Intro to 6.5

We want a function that will model a finite force that occurs over a very short period of time...

Start with:

$$d_\tau(t-c) = \begin{cases} 1/(2\tau) & \text{if } t \in [c-\tau, c+\tau] \\ 0 & \text{otherwise} \end{cases}$$



The *impulse* is the integral of force.

By design, d_τ has a unit impulse:

$$\int_{-\infty}^{\infty} d_\tau(t-c)\,dt = \frac{1}{2\tau}\int_{c-\tau}^{c+\tau} 1\,dt = 1$$

Definition: The Dirac Delta Function:

$$\delta(t-c) = \lim_{\tau\to 0} d_\tau(t-c)$$

Intuitively,

$$\delta(t-c) = \begin{cases} \text{''}\infty\text{''} & \text{if } t = c \\ 0 & \text{elsewhere} \end{cases}$$

However, $\delta(t-c)$ still has a *unit impulse*.

There are three main properties of $\delta(t-c)$:

- $\int_{-\infty}^{\infty} \delta(t-c)\,dt = 1$
- $\int_{-\infty}^{\infty} \delta(t-c)f(t)\,dt = f(c)$
- Lemma: $\delta(t-c)f(t) = \delta(t-c)f(c)$
- $\mathcal{L}(\delta(t-c)) = \int_{0}^{\infty} \mathrm{e}^{-st}\delta(t-c)\,dt = \mathrm{e}^{-cs}$

Let $y'(t) = \delta(t - c)$ with zero ICs. Solve for y using Laplace transforms.
SOLUTION:

$$sY = \mathrm{e}^{-cs} \quad \Rightarrow \quad Y = \frac{\mathrm{e}^{-cs}}{s} \quad \Rightarrow \quad y(t) = u_c(t)$$

Therefore,

> *"The 'derivative' of the Heaviside function is the Dirac function..."*

Mass-spring with zero ICs and strike a hammer (unit impulse) at times 0, 2π, 4π, and so on:

$$y'' + y = \delta(t) + \delta(t - 2\pi) + \delta(t - 4\pi) + \cdots = \sum_{n=0}^{\infty} \delta(t - 2n\pi)$$

SOLUTION:

$$(s^2 + 1)Y = \sum_{n=0}^{\infty} \mathrm{e}^{-2\pi ns} \quad \Rightarrow \quad Y = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-2\pi ns}}{s^2 + 1}$$

$$y(t) = \sum_{n=0}^{\infty} u_{2\pi n}(t) \sin(t - 2n\pi)$$

But, $\sin(t - 2n\pi) = \sin(t)$, so therefore:

$$y(t) = \sin(t) + u_{2\pi}(t)\sin(t) + u_{4\pi}(t)\sin(t) + u_{6\pi}(t)\sin(t) + \cdots$$

Writing this piecewise:

$$y(t) = \begin{cases} \sin(t) & \text{if } 0 \le t < 2\pi \\ 2\sin(t) & \text{if } 2\pi \le t < 4\pi \\ 3\sin(t) & \text{if } 4\pi \le t < 6\pi \\ \vdots & \end{cases}$$