From the HW in 3.3: Euler Equations

Suppose that we have a differential equation defined in terms of $y(t)$. Suppose we wish to make a change of variables so that y is a function of x, where

$$t = \mathrm{e}^x \quad \Rightarrow \quad \frac{dt}{dx} = \mathrm{e}^x = t$$

Find a formula for $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$:

By the chain rule, we could write:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} \cdot t = t\dot{y}$$

Similarly, to find a formula for the second derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt} \cdot t\right) = \frac{d}{dt}\left(\frac{dy}{dt} \cdot t\right) \cdot \frac{dt}{dx} = \left(\frac{d^2y}{dt^2} \cdot t + \frac{dy}{dt}\right) t = t^2\ddot{y} + t\dot{y}$$

Now, given a differential equation where y is a function of t:

$$t^2\ddot{y} + \alpha\, t\dot{y} + \beta\, y = 0$$

We rewrite it by adding/substracting $t\dot{y}$:

$$(t^2\dot{y} + t\dot{y}) + (\alpha - 1)t\dot{y} + \beta y = 0$$

Which can now be converted (and subsequently solved):

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0$$

Example (Exercise 38): Solve: $t^2\ddot{y} - 4t\dot{y} - 6y = 0$

SOLUTION: The differential equation in terms of x will be:

$$y'' + (-4 - 1)y' - 6y = 0 \quad \Rightarrow \quad y'' - 5y' - 6y = 0$$

This gives $r = 6, -1$. Therefore,

$$y_1 = \mathrm{e}^{6x} = (\mathrm{e}^x)^6 = t^6 \qquad y_2 = \mathrm{e}^{-x} = \frac{1}{\mathrm{e}^x} = \frac{1}{t} \quad \Rightarrow \quad y_1 = t^6 \qquad y_2 = \frac{1}{t}$$

and the solution to the original problem is:

$$y(t) = C_1 t^6 + \frac{C_2}{t}$$

In fact, we can summarize all the different solutions using the characteristic equation, just as we did before.

Summary: Euler Equations

Given $t^2y'' + \alpha t y' + \beta y = 0$ (derivative is with respect to t), the characteristic equation (for the DE in x) is:

$$r^2 + (\alpha - 1)r + \beta = 0$$

The solution depends on the kind of solutions we get from this. Note that instead of using $t = e^x$, we'll use $x = \ln(t)$ (and assume $t > 0$).

- Two real solutions, r_1, r_2. Then

  $$y_1(x) = e^{r_1 x} \quad \Rightarrow \quad y_1(t) = e^{r_1 \ln(t)} = t^{r_1}$$

  and

  $$y_2(x) = e^{r_2 x} \quad \Rightarrow \quad y_2(t) = e^{r_2 \ln(t)} = t^{r_2}$$

- Two complex solutions, $r = a \pm bi$. Then

  $$y_1 = e^{ax}\cos(bx) \quad \Rightarrow \quad y_1(t) = t^a \cos(\ln(t^b))$$

  and

  $$y_2 = e^{ax}\sin(bx) \quad \Rightarrow \quad y_2(t) = t^a \sin(\ln(t^b))$$

- One real r. In this case,

  $$y_1(x) = e^{rx} \quad \Rightarrow \quad y_1(t) = t^r$$

  $$y_2(x) = xe^{rx} \quad \Rightarrow \quad y_2(t) = t^r \ln(t)$$

Examples

1. Solve $t^2y'' + 5ty' + 4y = 0$

   SOLUTION: $r^2 + (5-1)r + 4 = 0$, so $(r+2)^2 = 0$, or $r = -2$ is a repeated root. The general solution is therefore

   $$y(t) = t^{-2}(C_1 + C_2 \ln(t))$$

2. Solve $t^2y'' - 6y = 0$

   SOLUTION: $r^2 + (0-1)r - 6 = 0$, or $(r+2)(r-3) = 0$. That yields

   $$y(t) = C_1 t^{-2} + C_2 t^3$$

3. Solve $t^2y'' - 3ty' + 8y = 0$.

   SOLUTION: $r^2 + (-3-1)r + 8 = 0$, so $r^2 - 4r + 8 = 0$, or $(r-2)^2 = -4$, so that $r = 2 \pm 2i$. The general solution is then

   $$y(t) = t^2(C_1 \cos(\ln(t^2)) + C_2 \sin(\ln(t^2)))$$