Selected Solutions, Section 5.1

In problems 1-14 even, use the Ratio Test to find the radius of convergence.

6. Use the Ratio Test:

$$\lim_{n\to\infty} \frac{|x-x_0|^{n+1}}{n+1} \cdot \frac{n}{|x-x_0|^n} = |x-x_0| \lim_{n\to\infty} \left(\frac{n}{n+1}\right) = |x-x_0|$$

The series converges absolutely if $|x-x_0| < 1$, and diverges if $|x-x_0| > 1$, so the radius is 1.

8. Use the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)!|x|^{n+1}}{(n+1)^{n+1}} \frac{n^n}{|x|^n n!} = |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

Do you recall the technique where we exponentiate to use L'Hospital's rule?

$$\left(\frac{n}{n+1}\right)^n = \mathrm{e}^{n\ln\left(\frac{n}{n+1}\right)}$$

so now we take the limit of the exponent:

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}$$

which is of the form 0/0. Continue with L'Hospital:

$$\lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{n+1}{n} \cdot \frac{n+1-n}{(n+1)^2}}{-\frac{1}{n^2}} == \lim_{n\to\infty} \frac{1}{n(n+1)} \cdot \frac{-n^2}{1} = \lim_{n\to\infty} \frac{-n}{n+1} = -1$$

Therefore,

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \mathrm{e}^{n\ln\left(\frac{n}{n+1}\right)} = \mathrm{e}^{-1}$$

And the ratio test:

$$\frac{|x|}{e} < 1 \quad \Rightarrow \quad |x| < \mathrm{e}$$

12. Actually, this is kind of a "trick question", although the usual procedure still works:

$$f(x) = x^2 \quad \Rightarrow \quad f(-1) = 1$$

$$f'(x) = 2x \quad \Rightarrow \quad f'(-1) = -2$$

$$f''(x) = 2 \quad \Rightarrow \quad f''(-1) = 2$$

Therefore,

$$x^2 = 1 - 2(x+1) + \frac{2}{2!}(x+1)^2 = 1 - 2(x+1) + (x+1)^2$$

(Notice that if you expand and simplify this, you get x^2 back.)

This is not an infinite series; no matter what x is, you can always add those three terms together: The radius of convergence is ∞.

14. At issue here is to find a pattern in the derivatives, so we can write the general form for the n^{th} derivative.

$$\begin{array}{lll} n=0 & f(x) = (1+x)^{-1} & f(0) = 1 \\ n=1 & f'(x) = -(1+x)^{-2} & f'(0) = -1 \\ n=2 & f''(x) = (-1)(-2)(1+x)^{-3} & f''(0) = 2 \\ n=3 & f'''(x) = (-1)(-2)(-3)(1+x)^{-4} & f'''(0) = -3! \end{array}$$

From this we see that:

$$f^{(n)}(0) = (-1)^n n!$$

The Taylor series (actually, the Maclaurin series) is:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} x^n = \sum_{n=0}^{\infty} (-x)^n$$

and this converges if $|x| < 1$ (its an alternating geometric series).

Alternatively, we could see this directly using the sum of the geometric series:

$$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

18. Given that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Compute y' and y'' by writing out the first four terms of each to get the general term. Show that, if $y'' = y$, then the coefficients a_0 and a_1 are arbitrary, and show the given recursion relation.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \ldots = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

If $y'' = y$, then the coefficients must match up, power by power:

$$a_0 = 2a_2 \qquad a_1 = 6a_3 \qquad a_2 = 12a_4 \qquad \ldots \qquad a_n = (n+2)(n+1)a_{n+2}$$

Problems 19-23 are some symbolic manipulation problems.

19. Rewrite the left side equation so that the powers of x match up.

20. Much the same. In this problem, we see that the first sum starts with a constant term, the second sum starts with x^1, and so does the sum on the left. Therefore, we would rewrite each sum to start with x^1 power:

$$\sum_{k=1}^{\infty} a_{k+1}x^k = a_1 + \sum_{n=1}^{\infty} a_{n+1}x^n$$

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1}x^n$$

Now each sum begins with the same power of x,

$$\sum_{k=1}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{n=1}^{\infty} a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = a_1 + \sum_{n-0}^{\infty} \left(a_{n+1} + a_{n-1}\right) x^n$$

21. You may use a different symbol for the summation index if you like (it is a dummy variable):

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

We would like this to be indexed using x^k, $k = 0, 1, 2, \ldots$. This means that $k = n - 2$ or $n = k + 2$. Making the substitutions in each term,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k$$

22. In this case, the powers begin with x^2, so we let $k = n + 2$ or $n = k - 2$, with $k = 2, 3, 4, \ldots$:

$$\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{k=2}^{\infty} a_{k-2}x^k$$

23. Take care of the product with x first,

$$x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k = \sum_{n=1}^{\infty} na_n x^n + \sum_{k=0}^{\infty} a_k x^k$$

The first sum could begin with zero- It would make the first term of the sum zero. Therefore,

$$\sum_{n=0}^{\infty} na_n x^n + \sum_{k=0}^{\infty} a_k x^k = \sum_{n=1}^{\infty} (n+1)a_n x^n$$