Selected Solutions, Section 5.2

For problems 2, 5, 6, 8 do not spend too much time finding the general term(s) of the series. The recurrence relations are typically as far as we'll need to go.
In each of these problems, we take:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

2. In this case,

$$y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

Our goal is to write this as a single sum with x^m in the general sum. The first and last sums above start with x^0, the middle one starts with x^1 power. However, we can simply start the sum at $n = 0$, because the first term would be 0.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

We substitute $m = n - 2$ in the first sum, and $m = n$ in the second and third:

$$\sum_{m=0}^{\infty} \left((m+2)(m+1) a_{m+2} - m a_m - a_m \right) x^m = 0$$

This represents a polynomial that is zero for all x, so the coefficients must be zero. This gives us the recursion relation (we simplified a bit as well):

$$(m+2)(m+1) a_{m+2} - (m+1) a_m = 0 \quad \Rightarrow \quad a_{m+2} = \frac{a_m}{m+2}$$

Notice that

$$a_2 = \frac{1}{2} a_0 \quad a_3 = \frac{1}{3} a_1 \quad a_4 = \frac{1}{4} a_2 = \frac{1}{4 \cdot 2} a_0 \quad a_5 = \frac{1}{5} a_3 = \frac{1}{5 \cdot 3} a_1$$

$$a_6 = \frac{1}{6} a_4 = \frac{1}{6 \cdot 4 \cdot 2} a_0 \quad a_7 = \frac{1}{7} a_5 = \frac{1}{7 \cdot 5 \cdot 3} a_1 \quad a_8 = \frac{1}{8} a_6 = \frac{1}{8 \cdot 6 \cdot 4 \cdot 2} a_0$$

and so on. We can write the solution $y(x)$ as:

$$y(x) = a_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{4 \cdot 2} x^4 + \frac{1}{6 \cdot 4 \cdot 2} x^6 + \ldots \right) + a_1 \left(x + \frac{1}{3} x^3 + \frac{1}{5 \cdot 3} x^5 + \frac{1}{7 \cdot 5 \cdot 3} x^7 + \ldots \right)$$

These two functions (in series form) make up our fundamental set. This is what we expect, since $y(0) = a_0$ and $y'(0) = a_1$ (our initial conditions).

5. Follows much the same procedure:

$$(1-x)y'' + y = (1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Multiply by the $1-x$ (distribute the sum) and incorporate the x into it:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Write this as a single sum involving x^m. The first sum starts with x^0, the second with x^1 and the third with x^0. We can do the same thing as before- Simply start the second sum at $n=1$ (the first term of the sum would be zero, so we're not changing it).

Now substitute $m=n-2$ in the first sum, $m=n-1$ into the second, and $m=n$ in the third.

$$\sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} - m(m+1)a_{m+1} + a_m\right] x^m = 0$$

Again, this is a polynomial set to zero, which means each coefficient must be zero:

$$(m+2)(m+1)a_{m+2} - m(m+1)a_{m+1} + a_m = 0 \qquad \text{for } m = 0, 1, 2, \ldots$$

Writing the equation as a recurrence relation:

$$a_{m+2} = \frac{m}{m+2}a_{m+1} - \frac{1}{(m+2)(m+1)}a_m$$

Therefore,

m	Recurrence		
$m=0$	$a_2 =$		$-\frac{1}{2}a_0$
$m=1$	$a_3 =$	$\frac{1}{3}a_2$	$-\frac{1}{6}a_1$
$m=2$	$a_4 =$	$\frac{1}{2}a_3$	$-\frac{1}{12}a_2$
$m=3$	$a_5 =$	$\frac{3}{5}a_4$	$-\frac{1}{20}a_3$

and so on. To get our fundamental set, solve these first with $a_0=1, a_1=0$, then with $a_0=0$, $a_1=1$ (factoring would be a little more work):

$$\begin{aligned} a_2 &= -\tfrac{1}{2} \\ a_3 &= -\tfrac{1}{6} \\ a_4 &= -\tfrac{1}{24} \end{aligned} \qquad \begin{aligned} a_2 &= 0 \\ a_3 &= -\tfrac{1}{6} \\ a_4 &= -\tfrac{1}{12} \\ a_5 &= -\tfrac{1}{24} \end{aligned}$$

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \ldots\right) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \ldots\right)$$

6. Goes much the same as Problem 5. Be sure to get your sums to all match in terms of beginning power of x and the index.

$$(2+x^2)y'' - xy' + 4y = (2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

Let's focus on that first sum: First, distribute the sum through the expression, then we'll try to make it work with the other terms of the sum (which both start with x^0).

$$(2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

The second sum begins with x^2, but we can shift the index from $n=2$ to $n=0$, since that would only add two 0 terms. Now we can collect all the sums together with the general term x^m:

$$\sum_{m=0}^{\infty} \left[2(m+2)(m+1)a_{m+2} + m(m-1)a_m - m a_m + 4a_m\right] x^m = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} + (n^2-2n+4)a_n\right] x^k$$

This gives us the recursion:

$$a_{n+2} = -\frac{n^2-2n+4}{2(n+2)(n+1)} a_n$$

or:

$$a_2 = -a_0 \quad a_3 = -\frac{1}{4}a_1 \quad a_4 = -\frac{1}{6}a_2 = \frac{1}{6}a_0$$

$$a_5 = -\frac{7}{40}a_3 = \frac{7}{160}a_1 \qquad a_6 = -\frac{1}{5}a_4 = -\frac{1}{30}a_0$$

and so on. Writing y in terms of its fundamental set (optional):

$$y(x) = a_0\left(1 - x^2 + \frac{1}{6}x^4 - \frac{1}{30}x^6 + \ldots\right) + a_1\left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \ldots\right)$$

8. Be careful in that our power series is now based at $x_0 = 1$ instead of $x_0 = 0$:

$$xy'' + y' + xy = x\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} + x\sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

The problem is that we cannot incorporate x into a series with an $(x-1)$ expansion. However, note that we can write

$$x = x - 1 + 1 \qquad \text{or } x = 1 + (x-1)$$

Making this substitution into the first sum,

$$(1+(x-1))\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1}$$

And similarly, into the last sum:

$$(1+(x-1))\sum_{n=0}^{\infty} a_n(x-1)^n = \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1}$$

Now we have 5 sums to make into one whose general term is $(x-1)^m$. Notice that our usual shift in the index won't work here- 2 of the sums start with $(x-1)^1$, the other 3 with $(x-1)^0$. We will re-write the sums so that we'll have a constant plus the sum, where necessary. Here are the five sums:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = 2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} = \sum_{n=1}^{\infty}(n+1)na_{n+1}(x-1)^n$$

$$\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = a_1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}(x-1)^n$$

$$\sum_{n=0}^{\infty} a_n(x-1)^n = a_0 + \sum_{n=1}^{\infty} a_n(x-1)^n$$

$$\sum_{n=0}^{\infty} a_n(x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1}(x-1)^n$$

Now simply collect it all into a single sum and extract the recursion- We really don't need to change to m in this case.

$$(2a_2 + a_1 + a_0) +$$

$$\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + (n+1)a_{n+1} + a_n + a_{n-1}\right](x-1)^n = 0$$

with recursion:

$$a_2 = -\frac{1}{2}(a_0 + a_1)$$

and for $n = 1, 2, 3, \ldots$ we have:

$$a_{n+2} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+2)(n-1)}$$

We could get the fundamental set of solutions in the usual way, but we can stop here.

15. I wanted you to work this through using our second technique for finding a series- That is, compute the derivatives directly (and then substitute them into the Taylor Series). In this case we want the first five non-zero terms- If the first 5 terms are non-zero, they would be:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4$$

Therefore, we only need to compute the derivatives from the DE: $y'' - xy' - y = 0$. Then we're given $y(0) = 2$, $y'(0) = 1$ and for the remaining terms:

$$y'' = xy' + y \quad \Rightarrow \quad y''(0) = 0 + y(0) = 2$$

$$y''' = y' + xy'' + y' = 2y' + xy'' \quad \Rightarrow \quad y'''(0) = 2y'(0) = 2$$

$$y^{(4)} = 2y'' + y'' + xy''' = 3y'' + xy''' \quad \Rightarrow \quad y^{(4)}(0) = 3y''(0) = 6$$

Therefore, the first five terms of the solution:

$$y(x) = 2 + x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 + \frac{6}{4!}x^4 + \cdots = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$$

Parts (b) and (c) need to be done on a computer (and are optional). For your convenience, I have plotted the 4 and 5 term expansions on the graph below using Maple.

If you've had Calc Lab, here is the Maple code I used:

```
Eqn15:=diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0;
inits:=y(0)=2, D(y)(0)=1;
IVP:={Eqn15,inits};
with(powseries):
f:=powsolve(IVP);
recursion_relation:=a(n)=subs(_k=n,f(_k));
F4:=tpsform(f,x,5);  #4th degree approx
F5:=tpsform(f,x,6); #5th degree approx
f4:=convert(F4,polynom,x);
f5:=convert(F5,polynom,x);
g:=rhs(dsolve(IVP,y(x)));
plot({g,f4,f5},x=-1..4,y=0..25);
```

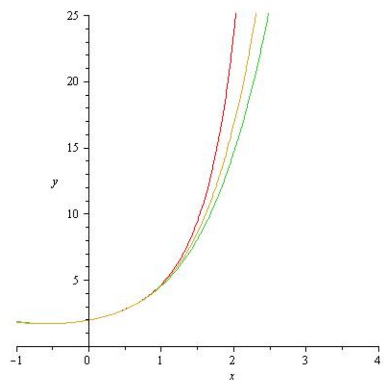

Figure 1: Order 4, 5, and the actual solution for Exercise 15.

And it gave me the plot in Figure 1.

16. Same technique as 15 - Directly compute the derivatives.

Given $(2+x^2)y'' - xy' + 4y = 0$ with $y(0) = -1$ and $y'(0) = 3$ Then :

$$(2+x^2)y'' - xy' + 4y = 0 \quad \Rightarrow \quad 2y''(0) - 0 + 4y(0) = 0 \quad \Rightarrow \quad y''(0) = -2y(0) = 2$$

For the third derivative, I would differentiate "in place" first, then simplify, then substitute $x = 0$:

$$(2+x^2)y''' + 2xy'' - xy'' - y' + 4y' = 0 \quad \Rightarrow (2+x^2)y''' + xy'' + 3y' = 0 \quad \Rightarrow \quad y'''(0) = -\frac{9}{2}$$

Similarly, after simplifying my next derivative:

$$(2+x^2)y^{(4)} + 3xy^{(3)} + 4y'' = 0 \quad \Rightarrow \quad y^{(4)}(0) = -2y''(0) = -4$$

Therefore, the first 5 terms of the solution is:

$$y(x) = -1 + 3x + \frac{2}{2!}x^2 - \frac{9}{2\cdot 3!}x^3 - \frac{4}{4!}x^4 + \cdots = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \cdots$$

(Again, parts (b) and (c) are optional since we need a computer) Using the same Maple commands (just changing the DE), we get the graph shown in Figure 2.

(**See Exercise 6 for the recurrence relation...**)

19. We make the suggested change of variables so that

$$x - 1 = t$$

Then change the differential equation from derivatives in x to derivatives in t:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt} \cdot 1$$

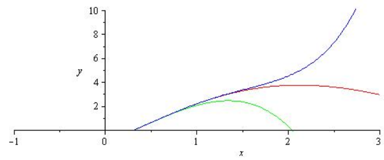

Figure 2: Order 4 (green) and 5 (blue) approximations and the solution (red) to the DE in Exercise 16. In this case, we'll need to increase the order of the solution by quite a bit in order to get better accuracy within this interval!

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{d^2y}{dt^2}$$

We also see that $(x-1)^2 = t^2$ and $(x^2-1) = ((t+1)^2-1) = t^2+2t$, so the DE becomes

$$y'' + t^2y' + (t^2+2t)y = 0$$

where the derivative is in t. Now we substitute the ansatz:

$$y = \sum_{n=0}^{\infty} a_n t^n \qquad \dot{y} = \sum_{n=1}^{\infty} n a_n t^{n-1} \qquad \dot{y} = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$$

we get (I'm bringing in t^2 and $2t$ into the last sum, so note that there are 4 sums total):

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+2} + \sum_{n=0}^{\infty} 2a_n t^{n+1} = 0$$

Checking these sums to make this a single polynomial, we see that the first power in each series: t^0, t^2, t^2, and t^1 respectively. We will make every sum begin with t^2. That means we need to break off the first two terms from the first sum:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = 2a_2 + 6a_3 t + \sum_{n=4}^{\infty} n(n-1)a_n t^{n-2}$$

And one term from the last sum:

$$\sum_{n=0}^{\infty} 2a_n t^{n+1} = 2a_0 t + \sum_{n=1}^{\infty} 2a_n t^{n+1}$$

Now, all sums begin with t^2. We'll try to fill in the following sum:

$$2a_2 + (6a_3 + 2a_0)t + \sum_{k=2}^{\infty}\left(\qquad\qquad\qquad \right)t^k = 0$$

For the first sum, substitute $k = n-2$ (or $n = k+2$). For the second, third and fourth sums, we substitute $k = n-1$ (or $n = k+1$). Doing this, we get:

$$2a_2 + (6a_3 + 2a_0)t + \sum_{k=2}^{\infty}\left((k+2)(k+1)a_{k+2} + (k-1)a_{k-1} + a_{k-1} + 2a_{k+1}\right)t^k = 0$$

Which we can simplify a bit:

$$2a_2 + (6a_3 + 2a_0)t + \sum_{k=2}^{\infty} \left((k+2)(k+1)a_{k+2} + ka_{k-1} + 2a_{k+1}\right) t^k = 0$$

Set the coefficients to 0, and we get that a_0, a_1 are free variables, and

$$a_2 = 0, \qquad a_3 = -\frac{1}{3}a_0, \qquad a_{k+2} = -\frac{1}{k+2}a_{k-1} - \frac{1}{(k+2)(k+1)}a_{k-2}$$

where the recurrence works for $k = 2, 3, 4, \cdots$ (we already have a_3). Therefore, computing a_4, a_5 in terms of a_0, a_1, we get:

$$a_4 = -\frac{1}{4}a_1 - \frac{1}{4\cdot 3}a_0$$

$$a_5 = -\frac{1}{5}a_2 - \frac{1}{5\cdot 4}a_1 = 0 - \frac{1}{5\cdot 4}a_1 = \frac{1}{20}a_1$$

$$a_6 = -\frac{1}{6}a_3 - \frac{1}{6\cdot 5}a_2 = \frac{1}{6\cdot 3}a_0$$

Therefore, factoring out a_0 and a_1, we get:

$$y = a_0\left(1 - \frac{1}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{18}t^6 + \cdots\right) + a_1\left(t - \frac{1}{4}t^4 + \frac{1}{20}t^5 + \cdots\right)$$

And we could get the desired result by back substituting $t = x - 1$ (to get a series solution in x).

20. Good practice with the Ratio Test. You should see that, for

$$y_1 = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2\cdot 3 \ldots (3n-1)(3n)}$$

the ratio of the $(n+1)^{\text{st}}$ term to the n^{th} term is:

$$\frac{|x|^3}{(3n+1)(3n+2)(3n+3)}$$

And for

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3\cdot 4 \ldots (3n)(3n+1)}$$

the ratio simplifies to:

$$\frac{|x|^3}{(3n+1)(3n+2)(3n+3)}$$

Thus, the limit of each is zero (so the radius of convergence is ∞).

23, 24 These are optional since they must be done in Maple.