3.1, Exercises 25, 27

25. Consider the DE below:

$$\mathbf{Y}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{Y}$$

(a) Check that the function $\mathbf{Y} = \begin{bmatrix} te^{2t} \\ -(t+1)e^{2t} \end{bmatrix}$ is a solution to the DE.

SOLUTION: We compute the derivative (be sure and use the product rule!), then compare it to $A\mathbf{Y}$ and see if we get the same thing:

- $\mathbf{Y}' = \begin{bmatrix} e^{2t} + 2te^{2t} \\ -e^{2t} - 2(t+1)e^{2t} \end{bmatrix} = \begin{bmatrix} (1+2t)e^{2t} \\ -(2t+3)e^{2t} \end{bmatrix}$
- $A\mathbf{Y} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} te^{2t} \\ -(t+1)e^{2t} \end{bmatrix} = \begin{bmatrix} te^{2t} + (t+1)e^{2t} \\ te^{2t} - 3(t+1)e^{2t} \end{bmatrix} = \begin{bmatrix} (1+2t)e^{2t} \\ -(2t+3)e^{2t} \end{bmatrix}$

(b) Solve the IVP, if $\mathbf{Y}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

We notice that, using our Y,

$$\mathbf{Y}(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

By the Linearity Principle, if Y solves our DE, so does cY for any constant c. In this case, if we let $c = -2$, then $-2\mathbf{Y}(t)$ gives the desired result.

27. Given the matrix A below, and the vectors $\mathbf{Y}_1, \mathbf{Y}_2$:

$$A = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} \qquad \mathbf{Y}_1(t) = \begin{bmatrix} e^{-3t} - 2e^{-4t} \\ e^{-3t} - 4e^{-4t} \end{bmatrix} \qquad \mathbf{Y}_2(t) = \begin{bmatrix} 2e^{-3t} + e^{-4t} \\ 2e^{-3t} + 2e^{-4t} \end{bmatrix}$$

(a) Check that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are solutions to the DE:

- $\mathbf{Y}_1' = \begin{bmatrix} -3e^{-3t} + 8e^{-4t} \\ -3e^{-3t} + 16e^{-4t} \end{bmatrix}$ and $\mathbf{Y}_2' = \begin{bmatrix} -6e^{-3t} - 4e^{-4t} \\ -6e^{-3t} - 8e^{-4t} \end{bmatrix}$
- $A\mathbf{Y}_1 = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} e^{-3t} - 2e^{-4t} \\ e^{-3t} - 4e^{-4t} \end{bmatrix} = \begin{bmatrix} -2e^{-3t} + 4e^{-4t} - e^{-3t} + 4e^{-4t} \\ 2e^{-3t} - 4e^{-4t} - 5e^{-3t} + 20e^{-4t} \end{bmatrix}$

  If you simplify this expression, you do get the expression for $\mathbf{Y}_1'$
  Similarly, for $\mathbf{Y}_2$:

  $$A\mathbf{Y}_2 = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 2e^{-3t} + e^{-4t} \\ 2e^{-3t} + 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -4e^{-3t} - 2e^{-4t} - 2e^{-3t} - 2e^{-4t} \\ 4e^{-3t} + 2e^{-4t} - 10e^{-3t} - 10e^{-4t} \end{bmatrix}$$

  If we simplify this, we get the expressions for $\mathbf{Y}_2$.

(b) Are the functions linearly independent? If not, they would be constant multiples of each other for all t. In particular, we can then check to see if that is the case at $t = 0$:

$$\mathbf{Y}_1(0) = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \qquad \mathbf{Y}_2(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

These are not along the same line (through the origin), so they are linearly independent.

(c) For the initial conditions, we want to find constants c_1 and c_2 so that

$$c_1\mathbf{Y}_1(0) + c_2\mathbf{Y}_2(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

This leads us to the system below, which we can solve via Cramer's Rule:

$$\begin{aligned} -c_1 + 3c_2 &= 2 \\ -3c_1 + 4c_2 &= 3 \end{aligned} \qquad \Rightarrow c_1 = \frac{\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} -1 & 3 \\ -3 & 4 \end{vmatrix}} = -\frac{1}{5} \qquad c_2 = \frac{\begin{vmatrix} -1 & 2 \\ -3 & 3 \end{vmatrix}}{5} = \frac{3}{5}$$

The solution is therefore

$$-\frac{1}{5}\mathbf{Y}_1(t) + \frac{3}{5}\mathbf{Y}_2(t)$$

(You can leave the answer in this form)