Example Questions, Exam 3, Math 244

1. Consider an unforced harmonic oscillator, and find the general solution if we have two complex roots to the characteristic equation.

   (a) For the rest of the questions, assume mass $m = 1$, damping factor $\gamma = 1$ and spring constant $k = 2$, and re-write the solution.

   (b) From which point forward will the exponential part of the solution be less than 1/10?

   (c) What is the natural (pseudo-)period?

   (d) If k is increased slightly, does the natural period increase or decrease? (You might look back at the general case).

2. Give the solution to the following initial value problems (IVP):

   (a) $u'' + 6u' + 9u = 0$ with $u(0) = 2$ and $u'(0) = 7/3$

   (b) $u'' + 6u' + 8u = 0$ with $u(0) = 1$ and $u'(0) = 0$

   (c) $2u'' + 3u = 0$ with $u(0) = 2$ and $u'(0) = -3$.

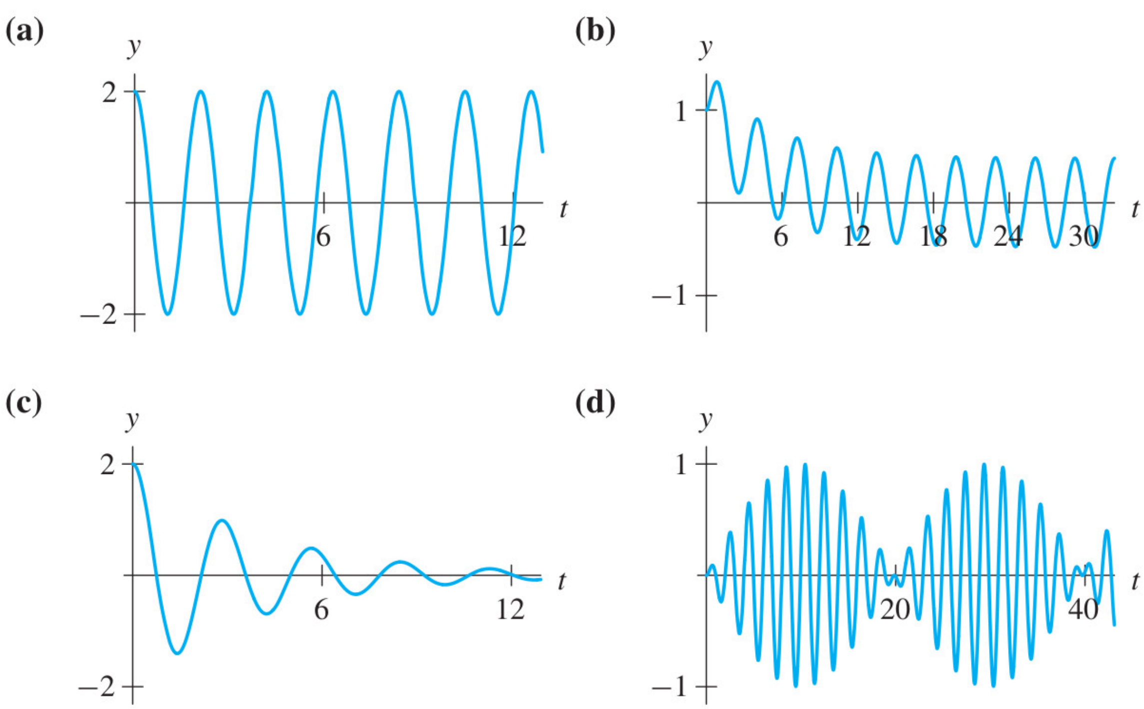
   (d) $u'' + 4u' + 5u = 0$ with $u(0) = 1$ and $u'(0) = 1$.

3. Find the general solution:

   (a) $y'' + 6y' + 8y = \mathrm{e}^{-t} + 3t^2 + \sin(3t)$

4. (Exercise 23 in Ch 4 review) Eight second order equations and four graphs are given below. For each **graph**, determine the differential equation for which $y(t)$ is a solution, and briefly state how you know your answer is correct. You should do this exercise without any technology.

(a) $y'' + 16y = 0$	(d) $2y'' - y' + 10y = 0$	(g) $y'' + 9y = 0$
(b) $y'' + 5y' + 5y = 5\cos(2t)$	(e) $2y'' + y' + 10y = 0$	
(c) $y'' + 5y' + y = 5\cos(4t)$	(f) $y'' + 3y = \cos(11t)$	(h) $y'' + 11y = \cos(3t)$



5. Rather than give you the second order linear differential equation, below are provided the solutions to the characteristic equation and the forcing function.

   Give the **form** of the solution to the particular part, but do NOT solve for the coefficients.

   (a) $\lambda = -1, -2$, $F(t) = t^2 + 3t$

   (b) $\lambda = -2 \pm i$, $F(t) = \mathrm{e}^{-2t}$

   (c) $\lambda = -2 \pm i$, $F(t) = t^2\mathrm{e}^{-2t}$

   (d) $\lambda = -2, -2$, $F(t) = t\mathrm{e}^{-2t}$

   (e) $\lambda = -1 \pm 8i$, $F(t) = \mathrm{e}^{-t}\cos(3t)$

6. Given $y'' + 2y' + 10y = \mathrm{e}^{-2t}\sin(2t)$, is there a way to "complexify" this problem? Solve the DE using the complexification.

7. Given $y'' + 2y' + y = \cos(2t) - 2\sin(2t)$, is there a way we can complexify the problem?

8. For the following, just find the amplitude and phase angle of the steady state solution (you should not need to find the general solution to the DE). You may use a calculator on these- on the exam I will make an extra effort to find nice numbers.

   (a) $y'' + 6y' + 13y = 2\cos(3t)$

   (b) $y'' + 2y' + 3y = \cos(2t)$

   (c) $y'' + 4y' + 4y = 2\cos(3t)$

9. Consider: $y'' + y' + 2y = \cos(\omega t)$.

   - Write the amplitude of the forced response in terms of ω.
   - Find ω that maximizes the amplitude.

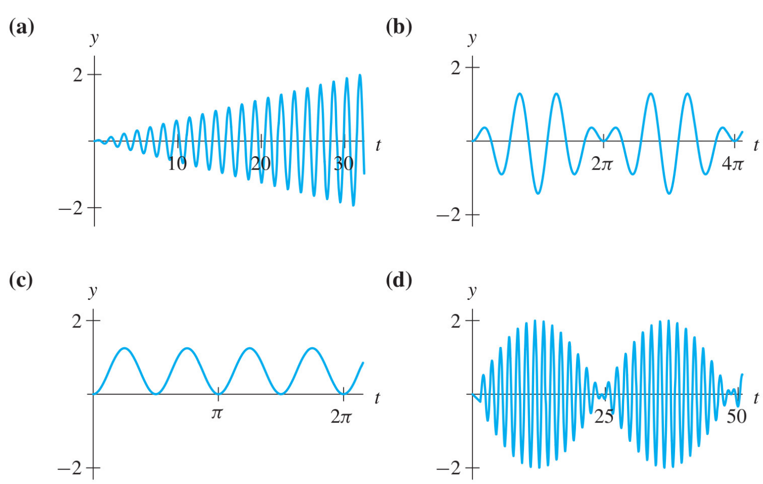
10. Go back to the differential equations in problem 2. Label each as either overdamped, underdamped, critically damped, or N/A.

11. Write $-\cos(3t) + \sqrt{3}\sin(3t)$ as $R\cos(\omega t - \delta)$.

12. Use complexification to find trig identities for $\cos(x - y)$ and $\sin(x - y)$.

13. Below are four graphs and six differential equations. Match each graph to its appropriate differential equation (this is Exercise 21, Section 4.3).

(a) $y'' + 16y = 10$	(c) $y'' + 16y = 5\cos(3t)$	(e) $y'' + 16y = \frac{1}{2}\cos(4t)$
(b) $y'' + 16y = -10$	(d) $y'' + 14y = 2\cos(4t)$	(f) $y'' + 2y' + 16y = \cos(4t)$



14. Suppose that two species, x and y are to be introduced to an island. It is known that the two species compete, but the precise nature of this interaction is unknown. We assume that $x(t)$ and $y(t)$ are modeled by a system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

If it is known that each species reproduces very slowly and there is intense competition between them, answer the following questions:

(a) What can we conclude about f_x and g_y at $(0,0)$?

(b) What can we conclude about f_y and g_x at $(0,0)$?

(c) Using the assumptions from the previous two answers, see if you can classify the origin using the Poincaré Diagram.

15. Short Answer:

(a) What is the extended linearity principle?

(b) What is "beating"? How do we find the period of a beat?

(c) What is "resonance"? (In both undamped and damped systems).

(d) Find all equilibrium solutions to $y'' + 4y = \sin(t)$.

(e) What is the frequency of the steady state solution to the equation:

$$y'' + 3y' + y = 4\cos(2t)$$

(f) Consider the following 3 equations, each of which has an equilibrium solution at the origin. Which two systems have phase portraits with the same local picture? Justify your answer.

(i) $\begin{aligned} x' &= 3\sin(x) + y \\ y' &= 4x + \cos(y) - 1 \end{aligned}$ (ii) $\begin{aligned} x' &= -3\sin(x) + y \\ y' &= 4x + \cos(y) - 1 \end{aligned}$ (iii) $\begin{aligned} x' &= -3\sin(x) + y \\ y' &= 4x + 3\cos(y) - 3 \end{aligned}$

16. For each system, (a) find and classify the equilibria, (b) sketch the nullclines, and (c) give a general sketch of the phase portrait.

(a) $\begin{aligned} x' &= x - 3y^2 \\ y' &= x - 3y - 6 \end{aligned}$

(b) $\begin{aligned} x' &= 10 - x^2 - y^2 \\ y' &= 3x - y \end{aligned}$

17. Consider the differential equation:

$$x'' + 2x' - 3x + x^3 = 0$$

(a) Convert this to a system of first order equations.

(b) Classify those equilibria using linearization.