Partial Fraction Decomposition (Summary)

Partial Fraction Decomposition is used when we have a fraction, $P(x)/Q(x)$, where P, Q are polynomials, and the degree of P is less than the degree of Q. **NOTE:** If the degree of the numerator is larger than the denominator, then perform long division first.

Assume Q is fully factored. We have 4 cases that we will consider.

Case I : Q has distinct linear factors,

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\ldots(a_kx + b_k)$$

Then:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_k}{a_kx + b_k}$$

Example (be sure you could have found the constants):

$$\frac{3x}{(x-1)(x-2)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{6}{x-2} - \frac{3}{x-1}$$

Case II : Q has some repeated linear factors. Let $a_1x + b_1$ be repeated r times. Then, instead of the single term $A_1/(a_1x + b_1)$, we have one term for each successive power in the denominator:

$$\frac{B_1}{a_1x + b_1} + \frac{B_2}{(a_1x + b_1)^2} + \ldots + \frac{B_r}{(a_1x + b_1)^r}$$

Example:

$$\frac{3x+1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} = -\frac{7}{x-1} - \frac{4}{(x-1)^2} + \frac{7}{x-2}$$

Case III : Q has some irreducible quadratic factors, not repeated. Let $ax^2 + bx + c$ be an irreducible quadratic factor for Q. Then the decomposition will have the term:

$$\frac{Ax + B}{ax^2 + bx + c}$$

Example:

$$\frac{3x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1} = \frac{3}{2}\frac{1}{x-1} + \frac{1}{2}\frac{-3x+3}{x^2+1}$$

Case IV : Q has some irreducible quadratic factors, some repeated. Suppose that $ax^2 + bx + c$ is a repeated quadratic factor (repeated r times). Then, instead of the single expression in Case III, we will have:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Example:

$$\frac{x+3}{(x-1)(x^2+1)^2} = \frac{1}{x-1} - \frac{x+1}{x^2+1} - \frac{2x+1}{(x^2+1)^2}$$

Worked Examples and Exercises

1. Worked Example:

$$\frac{6x^2 - 6x - 6}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$$

   Clear fractions:

$$6x^2 - 6x - 6 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$$

   And this statement must be true for all x. In particular, it must be true for $x = 1, x = -2$ and $x = 3$ (we chose these to zero out the others). Substituting, we get

$$A = 1 \qquad B = 2 \qquad C = 3$$

2. Worked Example:

$$\frac{x^2 - 2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2}$$

   Clear fractions. In this case, it might be best to solve for the coefficients in a slightly different manner- Equate the coefficients to the polynomials on the left and right:

$$x^2 - 2 = A(x^2+2) + (Bx + C)x = (A+B)x^2 + Cx + 2A$$

   so that:

$$1 = A + B, 0 = C, 2A = -2$$

   so: $A = -1, B = 2$ and $C = 0$:

$$\frac{x^2 - 2}{x(x^2+2)} = \frac{-1}{x} + \frac{2x}{x^2 + 2}$$

Exercises

For each of the following, first give the general form for the Partial Fraction expansion, then solve for the constants.

1. $\dfrac{x^2 + 1}{x^2 + 3x + 2}$
2. $\dfrac{2x + 3}{(x+1)^2}$
3. $\dfrac{4x^2 - 7x - 12}{x(x+2)(x-3)}$
4. $\dfrac{x^2 + 3}{x^3 + 2x}$
5. $\dfrac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}$
6. $\dfrac{3x^3 - x + 12}{x^2 - 1}$

Solutions

1. $\dfrac{x^2+1}{x^2+3x+2} = 1 - \dfrac{5}{x+2} + \dfrac{2}{x+1}$

2. $\dfrac{2x+3}{(x+1)^2} = \dfrac{2}{x+1} + \dfrac{1}{(x+1)^2}$

3. $\dfrac{4x^2-7x-12}{x(x+2)(x-3)} = \dfrac{2}{x} + \dfrac{9}{5}\,\dfrac{1}{x+2} + \dfrac{1}{5}\,\dfrac{1}{x-3}$

4. $\dfrac{x^2+3}{x^3+2x} = \dfrac{3}{2}\,\dfrac{1}{x} - \dfrac{1}{2}\,\dfrac{x}{x^2+2}$

5. $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)} = \dfrac{1}{x-1} - \dfrac{1}{(x-1)^2} - \dfrac{x-1}{x^2+1}$

6. $3x + \dfrac{7}{x-1} - \dfrac{5}{x+1}$