Extra Practice: Section 4.4

1. For each differential equation, find the general solution:

   (a) $y'' + 6y' + 9y = \mathrm{e}^{-2t}$
   SOLUTION: The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$, or $(\lambda + 3)^2 = 0$, so $\lambda = -3, -3$. The homogeneous part of the solution is therefore:

   $$y_h(t) = C_1\mathrm{e}^{-3t} + C_2 t\mathrm{e}^{-3t} = \mathrm{e}^{-3t}(C_1 + C_2 t)$$

   For the particular part, we guess: $y_p = \mathrm{e}^{-2t}$. Since this is not e^{-3t}, we can continue by putting it into the DE. Doing that, we get the equation:

   $$A\mathrm{e}^{-2t}(4 - 12 + 9) = \mathrm{e}^{-2t} \quad \Rightarrow \quad A = 1$$

   The general solution is therefore

   $$y(t) = \mathrm{e}^{-3t}(C_1 + C_2 t) + \mathrm{e}^{-2t}$$

   (b) $y'' + 6y' + 9y = \mathrm{e}^{-3t}$
   SOLUTION: We've already gotten the homogeneous part of the solution, so we just analyze the particular part, with $y_p = A\mathrm{e}^{-3t}$. But this is part of the homogeneous solution, so we multiply by t: $y_p = At\mathrm{e}^{-3t}$. But this is still part of the homogeneous solution, so multiply by t again to finally get

   $$y_p = At^2\mathrm{e}^{-3t} \qquad y_p' = 2At\mathrm{e}^{-3t} - 3At^2\mathrm{e}^{-3t} \qquad y_p'' = 2A\mathrm{e}^{-3t} - 12At\mathrm{e}^{-3t} + 9At^2\mathrm{e}^{-3t}$$

   Line these up to make the computations easier:

   $$\begin{array}{rlll} y_p'' & = 2A\mathrm{e}^{-3t} & -12At\mathrm{e}^{-3t} & +9At^2\mathrm{e}^{-3t} \\ 6y_p' & = & 12At\mathrm{e}^{-3t} & -18At^2\mathrm{e}^{-3t} \\ +9y_p & = & & 9At^2\mathrm{e}^{-3t} \\ \hline \mathrm{e}^{-2t} & = 2A\mathrm{e}^{-3t} & & \end{array}$$

   The full solution is therefore

   $$y(t) = \mathrm{e}^{-3t}\left(C_1 + C_2 t + \frac{1}{2}t^2\right)$$

   (c) $y'' + 6y' + 9y = \cos(3t)$
   SOLUTION: We first complexify the problem to $y'' + 6y' + 9y = \mathrm{e}^{3it}$, and so we want the real part of the particular solution.
   In this case, we guess that $y_p = A\mathrm{e}^{3it}$, and subsitute it back in as usual. I like to factor out the common terms as I go:

   $$A\mathrm{e}^{3it}(-9 + 6(3i) + 9) = \mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{1}{18i} = -\frac{i}{18}$$

Now we want the real part of Ae^{3it}:

$$Ae^{3it} = \frac{-i}{18}\cos(3t) + \frac{1}{18}\sin(3t)$$

so that the full solution is:

$$y(t) = e^{-3t}(C_1 + C_2 t) + \frac{1}{18}\sin(3t)$$

(d) $y'' + 6y' + 9y = \sin(3t)$

SOLUTION: See the previous problem, as the computations will be identical. The only difference is that we want the imaginary part of y_p, so the full solution is:

$$y(t) = e^{-3t}(C_1 + C_2 t) - \frac{3}{18}\cos(3t)$$

2. Find the particular part of the solution to the following DE:

$$y'' + py' + qy = \cos(\omega t)$$

SOLUTION: Let $y_p = Ae^{i\omega t}$, and subsitute into the DE. We will want the real part of what we get. Note the derivatives are:

$$y_p = Ae^{i\omega t} \qquad y_p' = i\omega Ae^{i\omega t} \qquad y_p'' = -\omega^2 Ae^{i\omega t}$$

If we factor $Ae^{i\omega t}$ out of the expression as we go, we end up with:

$$Ae^{i\omega t}(-\omega^2 + ip\omega + q) = e^{i\omega t} \quad \Rightarrow \quad A = \frac{1}{(q-\omega^2) + ip\omega}$$

Now multiply $Ae^{i\omega t}$ out, and you should find that the real part is:

$$y_p(t) = \frac{q-\omega^2}{(q-\omega^2)^2 + p^2\omega^2}\cos(\omega t) + \frac{pw}{(q-\omega^2)^2 + p^2\omega^2}\sin(\omega t)$$

3. Suppose that $Y = Ae^{i\omega t} = \dfrac{1}{a+ib}e^{i\omega t}$.

Show that the amplitude of the real part of Y is $1/|a+ib|$, and the phase shift (δ) of the real part of Y is the same as the polar angle for $a+ib$.

HINT: Multiply out the expression for Y first.

SOLUTION: If we multiply the expression out, we can get the real part of Y. Note that

$$\frac{1}{a+ib} = \frac{a}{a^2+b^2} - i\,\frac{b}{a^2+b^2}$$

Here we go:

$$Y=\left(\frac{a}{a^2+b^2}-i\,\frac{b}{a^2+b^2}\right)(\cos(\omega t)+i\sin(\omega t))=$$

$$\left(\frac{a}{a^2+b^2}\cos(\omega t)+\frac{b}{a^2+b^2}\sin(\omega t)\right)+i\left(-\frac{b}{a^2+b^2}\cos(\omega t)+\frac{a}{a^2+b^2}\sin(\omega t)\right)$$

The real part of Y is the first part of the last expression. Writing this expression as $R\cos(\omega t-\delta)$, we find

$$R=\sqrt{\frac{a^2}{(a^2+b^2)^2}+\frac{b^2}{(a^2+b^2)^2}}=\sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}}=\sqrt{\frac{1}{a^2+b^2}}=\frac{1}{\sqrt{a^2+b^2}}=\frac{1}{|a+ib|}$$

For the angle δ, we take:

$$\delta=\arctan\left(\frac{b}{a}\right)$$

where we add π if (a,b) is in Quadrant II or III. This is the same computation we would make for the polar angle of $a+ib$.

4. Find the amplitude and phase angle for the solution to the following DEs. Try to do as little work as you can, and you may use a calculator to assist you, if necessary.

   (a) $y''+y'+4y=\cos(2t)$
   SOLUTION:
   Let $y_p=A\mathrm{e}^{2it}$ and subsitute it into the DE. We then get:

   $$A\mathrm{e}^{2ti}\,(-4+2i+4)=\mathrm{e}^{2it}\qquad\Rightarrow\qquad A=\frac{1}{2i}$$

   We can get the amplitude and phase angle from $2i$: $R=1/|2i|=1/2$ and $\delta=\frac{\pi}{2}$
   Side Remark: Looking at the solution we get, we see that

   $$\frac{-i}{2}(\cos(2t)+i\sin(2t))=\frac{1}{2}\sin(2t)-\frac{i}{2}\cos(2t)$$

   so $y_p(t)=\frac{1}{2}\sin(2t)=\frac{1}{2}\cos(2t-\pi/2)$

   (b) $y''+\frac{1}{2}y'+2y=\cos(2t)$
   SOLUTION: Same idea as before. Let $y_p=A\mathrm{e}^{2it}$ and put it into the DE to get:

   $$A\mathrm{e}^{2it}\,(-4+i+2)=\mathrm{e}^{2it}\qquad\Rightarrow\qquad A=\frac{1}{-2+i}$$

   The amplitude of the response is therefore:

   $$\frac{1}{|-2+i|}=\frac{1}{\sqrt{4+1}}=\frac{1}{\sqrt{5}}$$

The phase angle is the phase angle of $-2+i$ (add π to the arctangent since we're in Quadrant II):

$$\delta = \tan^{-1}\left(-\frac{1}{2}\right) + \pi$$

Side Remark: We want the real part of

$$\frac{1}{-2+i}(\cos(2t) + i\sin(2t)) = \left(\frac{-2}{5} - \frac{1}{5}i\right)(\cos(2t) + i\sin(2t))$$

which is:

$$-\frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t)$$

which has amplitude and phase shift (as $R\cos(\omega t - \delta)$:

$$R = \sqrt{\frac{4+1}{25}} = \frac{1}{\sqrt{5}} \qquad \delta = \tan^{-1}\left(-\frac{1}{2}\right) + \pi$$

which is what we computed above without actually having to compute $y_p(t)$.

5. Given $y'' + \frac{1}{2}y' + 2y = \cos(\omega t)$, find the value of ω that gives the maximum amplitude.

   SOLUTION: Let $y_p = Ae^{i\omega t}$. Then substituting it into the DE, we get

$$A\left(-\omega^2 + i\frac{1}{2}\omega + 2\right) = 1 \quad \Rightarrow \quad A = \frac{1}{(2-\omega^2) + i\omega/2}$$

   The magnitude of A is then given by:

$$|A| = \frac{1}{\sqrt{(2-\omega^2)^2 + \omega^2/4}} = \frac{1}{\sqrt{F(\omega)}}$$

   As we showed in class, the critical points of $|A|$ are found by $F'(\omega) = 0$, or:

$$2(2-\omega^2)^1(-2\omega) + \omega/2 = 0$$

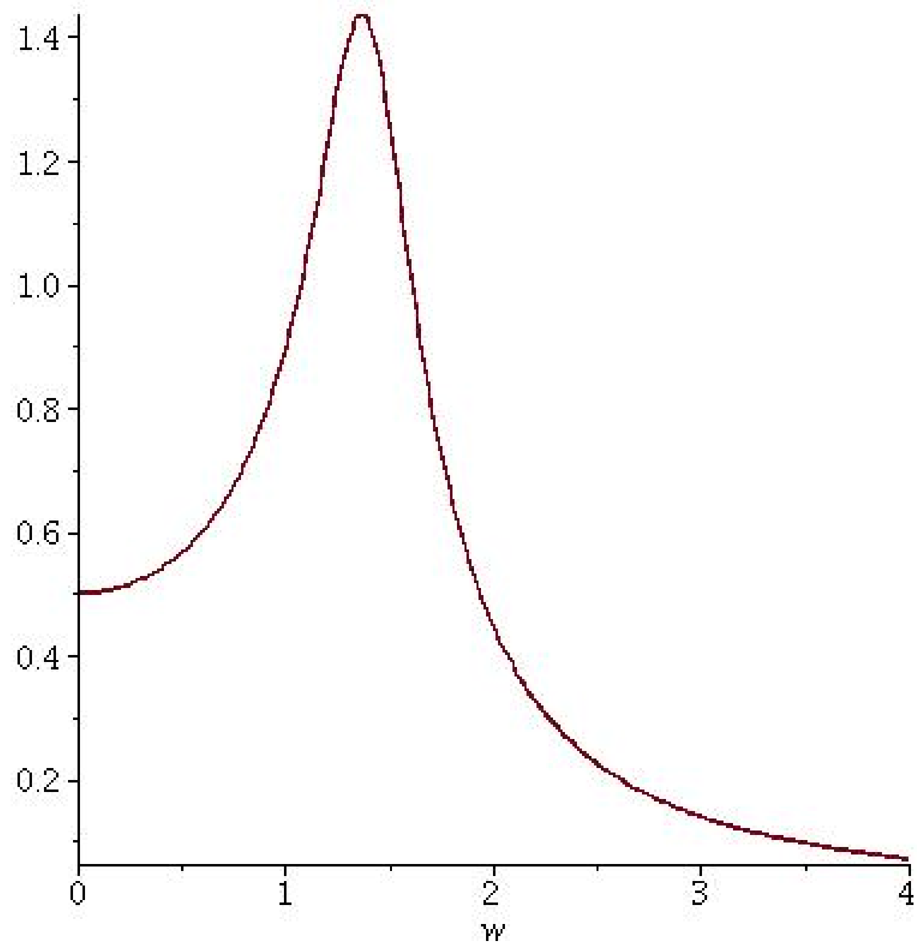
   Since $\omega \neq 0$, we can divide it out and get:

$$-8 + 4\omega^2 + \frac{1}{2} = 0 \quad \Rightarrow \quad 4\omega^2 = \frac{15}{2} \quad \Rightarrow \quad \omega^2 = \frac{15}{8}$$

   We'll only take $\omega > 0$, giving

$$\omega = \sqrt{\frac{15}{8}} \approx 1.37$$

   Just for completeness, the graph of $|A| = \frac{1}{\sqrt{F(\omega)}}$ is shown below.


1.4
1.2
1.0
0.8
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0.4
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1
2
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4
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