Overview of Complex Numbers

Definition 1 *The* **complex number** z *is defined as:* $z = a+bi$*, where* a*,* b *are real numbers and* $i = \sqrt{-1}$.

General notes about $z = a + bi$

- Engineers typically use j instead of i.
- Examples of complex numbers: $5 + 2i, \quad 3 - \sqrt{2}i, \quad 3, \quad -5i$
- Powers of i are cyclic: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$ and so on. Notice that the cycle is: $i, -1, -i, 1$, then it repeats.
- All real numbers are also complex (by taking $b = 0$), so the set of real numbers is a subset of the complex numbers.

We can split up a complex number by using **the real part** and **the imaginary part** of the number z:

Definition 2 *The* **real part** *of* $z = a + bi$ *is* a*, or in notation we write:*
$\mathrm{Re}(z) = \mathrm{Re}(a + bi) = a$
The **imaginary part** *of* $a + bi$ *is* b*, or in notation we write:*
$\mathrm{Im}(z) = \mathrm{Im}(a + bi) = b$

The same definitions can be applied to vectors that contain complex numbers. For example, if $\mathbf{x}$ is the vector below, we can compute the real and imaginary parts of $\mathbf{x}$ as shown:

$$\mathbf{x} = \begin{bmatrix} 1+3i \\ 2-i \end{bmatrix} \quad \Rightarrow \quad \mathrm{Re}(\mathbf{x}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathrm{Im}(\mathbf{x}) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

1 Visualizing Complex Numbers

To visualize a complex number, we use the complex plane $\mathbb{C}$, where the horizontal (or x-) axis is for the real part, and the vertical axis is for the imaginary part. That is, $a + bi$ is plotted as the point (a, b).

In Figure 1, we can see that it is also possible to represent the point $a + bi$, or (a, b) in **polar form**, by computing its modulus (or size) r, and angle (or argument) θ as:

$$r = |z| = \sqrt{a^2 + b^2} \qquad \theta = \arg(z)$$

Once we do that, we can write:

$$z = a + bi = r(\cos(\theta) + i\sin(\theta))$$

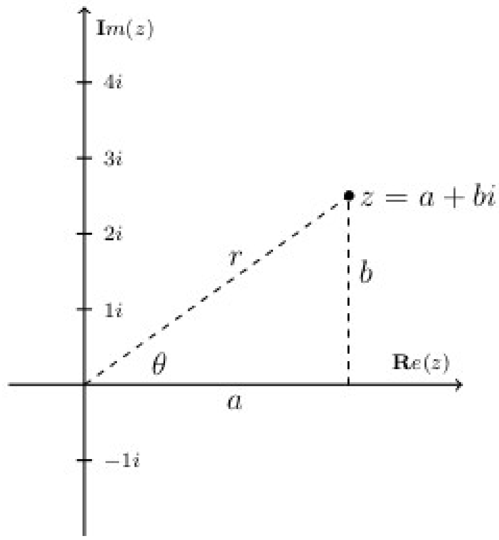

Figure 1: Visualizing $z = a + bi$ in the complex plane. Shown are the modulus (or length) r and the argument (or angle) θ, so $z = r(\cos(\theta) + i\sin(\theta))$.

We have to be a bit careful defining θ. For example, just adding a multiple of 2π will yield an equivalent number for θ. Typically, θ is defined to be the 4-quadrant "inverse tangent"[1] that returns $-\pi < \theta \leq \pi$.

That is, formally we can define the argument as:

$$\theta = \arg(a+bi) = \begin{cases} \tan^{-1}(b/a) & \text{if } a > 0 & \text{(Quad I and IV)} \\ \tan^{-1}(b/a) + \pi & \text{if } a < 0 \text{ and } b \geq 0 & \text{(Quad II)} \\ \tan^{-1}(b/a) + \pi & \text{if } a < 0 \text{ and } b < 0 & \text{(Quad III)} \\ \pi/2 & \text{if } a = 0 \text{ and } b > 0 & \text{(Upper imag axis)} \\ -\pi/2 & \text{if } a = 0 \text{ and } b < 0 & \text{(Lower imag axis)} \\ \text{Undefined} & \text{if } a = 0 \text{ and } b = 0 & \text{(The origin)} \end{cases}$$

This may look confusing, but it is simple- Always locate the point you are converting on the complex plane. Your calculator will only return angles in Quadrants I and IV, so if your point is not in one of those, add π. The exception to the rule is division by zero, but these points are easy to locate in the plane.

Examples

Find the modulus r and argument θ for the following numbers, then express z in polar form:

- $z = -3$:

  SOLUTION: $r = 3$ and $\theta = \pi$ so $z = 3(\cos(\pi) + i\sin(\pi))$

[1]For example, in Maple this special angle is computed as `arctan(b,a)`, and in Matlab the command is `atan2(b,a)`.

- $z = 2i$:

  SOLUTION: $r = 2$ and $\theta = \pi/2$ so $z = 2(\cos(\pi/2) + i\sin(\pi/2))$

- $z = -1 + i$:

  SOLUTION: $r = \sqrt{2}$ and $\theta = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ so

$$z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

- $z = -3 - 2i$ (Numerical approx from Calculator OK):

  SOLUTION: $r = \sqrt{14}$ and $\theta = \tan^{-1}(2/3) - \pi \approx 0.588 - \pi \approx -2.55$ rad, or

$$z = \sqrt{14}\,(\cos(-2.55) + i\sin(-2.55)) = \sqrt{14}(\cos(2.55) - i\sin(2.55))$$

  Note: We used the "even" symmetry of the cosine and the "odd" symmetry of the sine to do the simplification:

$$\cos(-x) = \cos(x) \qquad \text{and} \qquad \sin(-x) = -\sin(x)$$

2 Operations on Complex Numbers

2.1 The Conjugate of a Complex Number

If $z = a + bi$ is a complex number, then its *conjugate*, denoted by $\bar{z}$ is $a - bi$. For example,

$$z = 3 + 5i \Rightarrow \bar{z} = 3 - 5i \qquad z = i \Rightarrow \bar{z} = -i \qquad z = 3 \Rightarrow \bar{z} = 3$$

Graphically, the conjugate of a complex number is it's mirror image across the horizontal axis. If z has magnitude r and argument θ, then $\bar{z}$ has the same magnitude with a negative argument.

Example

If $z = 3(\cos(\pi/2) + i\sin(\pi/2))$, find the conjugate $\bar{z}$:

$$\bar{z} = 3(\cos(-\pi/2) + i\sin(-\pi/2)) = 3(\cos(\pi/2) - i\sin(\pi/2))$$

2.2 Add/Subtract, Multiply/Divide, Modulus or length

To add (or subtract) two complex numbers, add (or subtract) the real parts and the imaginary parts separately. This is like adding polynomials (with i in place of x):

$$(a + bi) \pm (c + di) = (a + c) \pm (b + d)i$$

To multiply, expand it as if you were multiplying polynomials, with i in place of x:

$$(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac-bd) + (ad+bc)i$$

and simplify using $i^2 = -1$.

Division by complex numbers $\frac{z}{w}$, is defined by translating it to real number division by rationalizing the denominator- multiply top and bottom by the conjugate of the denominator:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Example:

$$\frac{1+2i}{3-5i} = \frac{(1+2i)(3+5i)}{(3-5i)(3+5i)} = \frac{(1+2i)(3+5i)}{3^2+5^2} = \frac{-7}{34} + \frac{11}{34}i$$

The length (or modulus) of a complex number is denoted by $|z|$, and using the triangle and polar form, we've seen that:

$$z = a + ib \quad \Rightarrow \quad |z| = \sqrt{a^2+b^2}$$

There is another way to compute this using the conjugate which is often handy:

$$z\bar{z} = (a+bi)(a-bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2 = |z|^2$$

3 The Polar Form of Complex Numbers

The polar form of a complex number,

$$z = r\cos(\theta) + ir\sin(\theta)$$

has a beautiful counterpart using the complex exponential function, $\mathrm{e}^{i\theta}$. First, we'll define it using Euler's formula (although it is possible to *prove* Euler's formula).

Definition (Euler's Formula): $\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta)$.

> We can now express the polar form of a complex number slightly differently:
>
> $$z = r\mathrm{e}^{i\theta} \quad \text{where} \quad r = |z| = \sqrt{a^2+b^2} \qquad \theta = \arg(z)$$

An important note about this expression: The rules of exponentiation still apply in the complex case. For example,

$$\mathrm{e}^{a+ib} = \mathrm{e}^a\mathrm{e}^{ib} \qquad \text{and} \qquad \mathrm{e}^{i\theta}\mathrm{e}^{i\beta} = \mathrm{e}^{(\theta+\beta)i} \qquad \text{and} \qquad \left(\mathrm{e}^{i\theta}\right)^n = \mathrm{e}^{in\theta}$$

Here are some numerical examples of converting to and from polar form using Euler's Formula.

Examples

Given the complex number in $a + bi$ form, give the polar form, and vice-versa:

1. $z = 1 + i$

   SOLUTION: $r = \sqrt{2}$ and $\theta = \pi/4$, so $z = \sqrt{2}\mathrm{e}^{i\pi/4}$.

2. $z = 2i$

   SOLUTION: Since $r = 2$ and $\theta = \pi/2$, $z = 2\mathrm{e}^{i\pi/2}$

3. $z = 2\mathrm{e}^{-i\pi/3}$

   SOLUTION: We recall that $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$, so

$$z = 2(\cos(-\pi/3) + i\sin(-\pi/3)) = 2(\cos(\pi/3) - i\sin(\pi/3)) = 1 - \sqrt{3}\,i$$

4 Exponentials and Logs

The logarithm of a complex number is easy to compute if the number is in polar form. We use the normal rule of logs: $\ln(ab) = \ln(a) + \ln(b)$, or in the case of polar form:

$$\ln(a + bi) = \ln\left(r\mathrm{e}^{i\theta}\right) = \ln(r) + \ln\left(\mathrm{e}^{i\theta}\right) = \ln(r) + i\theta$$

Where we leave the last step as intuitively clear, but we don't prove it here (we have to be careful about the choice of θ as described earlier).

The logarithm of zero is left undefined (as in the real case). However, we can now compute things like the log of a negative number!

$$\ln(-1) = \ln\left(1 \cdot \mathrm{e}^{i\pi}\right) = i\pi \qquad \text{or the log of } i: \quad \ln(i) = \ln(1) + \frac{\pi}{2}i = \frac{\pi}{2}i$$

To exponentiate a number, we convert it to multiplication (a trick we used in Calculus when dealing with things like x^x):

$$a^b = \mathrm{e}^{b\ln(a)}$$

Examples of Exponentiation

- $2^i = \mathrm{e}^{i\ln(2)} = \cos(\ln(2)) + i\sin(\ln(2))$
- $\sqrt{1+i} = (1+i)^{1/2} = \left(\sqrt{2}\mathrm{e}^{i\,\pi/4}\right)^{1/2} = (2^{1/4})\mathrm{e}^{i\,\pi/8}$
- $i^i = \mathrm{e}^{i\ln(i)} = \mathrm{e}^{i(i\pi/2)} = \mathrm{e}^{-\pi/2}$

5 Real Polynomials and Complex Numbers

If $ax^2 + bx + c = 0$, then the solutions come from the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the past, we only took real roots. Now we can use complex roots. For example, the roots of $x^2 + 1 = 0$ are $x = i$ and $x = -i$.

Check:

$$(x - i)(x + i) = x^2 + xi - xi - i^2 = x^2 + 1$$

Some facts about polynomials when we allow complex roots:

1. An n^{th} degree polynomial can always be factored into n roots. (Unlike if we only have real roots!) This is the *Fundamental Theorem of Algebra*.

2. If $a+bi$ is a root to a real polynomial, then $a-bi$ must also be a root. This is sometimes referred to as "roots must come in conjugate pairs".

6 Exercises

1. Suppose the roots to a cubic polynomial are $a = 3$, $b = 1-2i$ and $c = 1+2i$. Compute $(x-a)(x-b)(x-c)$.

2. Find the roots to $x^2 - 2x + 10$. Write them in polar form.

3. Show that:
$$\text{Re}(z) = \frac{z + \bar{z}}{2} \qquad \text{Im}(z) = \frac{z - \bar{z}}{2i}$$

4. For the following, let $z_1 = -3 + 2i$, $z_2 = -4i$
   (a) Compute $z_1\bar{z}_2$, z_2/z_1
   (b) Write z_1 and z_2 in polar form.

5. In each problem, rewrite each of the following in the form $a + bi$:
   (a) e^{1+2i}
   (b) e^{2-3i}
   (c) $\mathrm{e}^{i\pi}$
   (d) 2^{1-i}
   (e) $\mathrm{e}^{2-\frac{\pi}{2}i}$
   (f) π^i

6. For fun, compute the logarithm of each number:
   (a) $\ln(-3)$
   (b) $\ln(-1+i)$
   (c) $\ln(2\mathrm{e}^{3i})$