## Math 244 Extra Practice: Conversions

We stopped our consideration of second order DEs- We'll re-visit it later, but we want to pause for a moment to show that all  $n^{\text{th}}$  order differential equations can be converted to a system of first order DEs. We can also go backwards for some systems- Converting a system to an  $n^{\text{th}}$  order DE.

We'll show you how to do the conversions via some examples.

1. Convert y'' = ty' + 3y into a system of first order.

SOLUTION: We introduce new variables- Let  $x_1 = y$  and  $x_2 = y'$ . We write a system using  $x_1$  and  $x_2$  (note that  $x_1, x_2$  are functions of t)

$$\begin{array}{rcl} x'_1 &= y' = x_2 \\ x'_2 &= y'' = ty' + 3y = 3x_1 + tx_2 \end{array} \Rightarrow \begin{array}{rcl} x'_1 &= x_2 \\ x'_2 &= 3x_1 + tx_2 \end{array}$$

(Notice that the answer should be all in terms of the new variables)

2. Convert 2y'' + 4y' + 3y = 0 into a system of first order DEs:

SOLUTION: We introduce new variables- Let  $x_1 = y$  and  $x_2 = y'$ . We write a system using  $x_1$  and  $x_2$  (note that  $x_1, x_2$  are functions of t)

$$\begin{array}{rcl} x_1' &= y' = x_2 \\ x_2' &= y'' = -\frac{3}{2}y - 2y' = -\frac{3}{2}x_1 - 2x_2 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x_1' &= x_2 \\ x_2' &= -\frac{3}{2}x_1 - 2x_2 \end{array}$$

3. Convert  $y''' = 3y'' - 2y' + y + t^2$  into a system of first order DEs.

SOLUTION: We'll need three variables now: Let  $x_1 = y$ ,  $x_2 = y'$  and  $x_3 = y''$ . Then forming the system of differential equations proceeds much as before:

$$\begin{array}{ll} x_1' &= y' = x_2 & & & x_1' &= x_2 \\ x_2' &= y'' = x_3 & & \Rightarrow & x_2' &= x_3 \\ x_3' &= y''' = 3y'' - 2y' + y + t^2 = 3x_3 - 2x_2 + x_1 + t^2 & & x_3' &= 3x_3 - 2x_2 + x_1 + t^2 \end{array}$$

4. Convert  $y^{(iv)} = y + y' + y'' + y'''$  to a system of first order DEs. SOLUTION: We need 4 variables:  $x_1, x_2, x_3, x_4$ , and we end up with:

$$\begin{array}{ll} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= x_1 + x_2 + x_3 + x_4 \end{array}$$

## Going Backwards: A system to second order

Given a linear system with constant coefficients, convert to an equivalent second order linear homogeneous DE.

The idea will be that the first equation will be used to substitute into the second so that we end up with an equation all in  $x_2$  (or equivalently, all in  $x_1$ ).

$$\begin{array}{ll} x_{1}' &= ax_{1} + bx_{2} \\ x_{2}' &= cx_{1} + dx_{2} \end{array} \Rightarrow \quad x_{2} = \frac{1}{b}x_{1}' - \frac{a}{b}x_{1} \\ \end{array}$$

Now use the second equation and substitute:

$$x'_{2} = cx_{1} + dx_{2} \quad \Rightarrow \quad (\frac{1}{b}x'_{1} - \frac{a}{b}x_{1})' = cx_{1} + (\frac{1}{b}x'_{1} - \frac{a}{b}x_{1})$$

Notice that this is all in terms of  $x_1$ . Here are a couple of numerical examples:

1. Convert to an equivalent 2d order DE:  $\begin{array}{cc} x_1' &= 3x_1 + x_2 \\ x_2' &= -x_1 + 2x_2 \end{array}$ 

SOLUTION: Using the first equation, we have  $x_2 = x'_1 - 3x_1$ . Substitute into the second to get:

$$(x_1' - 3x_1)' = -x_1 + 2(x_1' - 3x_1) \quad \Rightarrow \quad x_1'' - 3x_1' = -x_1 + 2x_1' - 6x_1$$

Simplifying, we get

$$x_1'' - 5x_1' + 7x_1 = 0$$

And this is our second order DE. As a "side remark", if I had used the second equation to substitute into the first equation, I would have ended up with the same second order DE (in  $x_2$  instead of  $x_1$ ).

2. Convert to an equivalent 2d order DE:  $\begin{array}{c} x_1' = x_1 + \frac{1}{2}x_2 \\ x_2' = 2x_1 - 3x_2 \end{array}$ 

SOLUTION: Using the first equation, we have  $x_2 = 2x'_1 - 2x_1$ . Substitute into the second to get:

$$(2x_1' - 2x_1)' = 2x_1 - 3(2x_1' - 2x_1) \quad \Rightarrow \quad 2x_1'' - 2x_1' = 2x_1 - 6x_1' + 6x_1$$

Simplifying, we get

$$2x_1'' + 4x_1' - 8x_1 = 0$$

And this is our second order DE.

## Exercises

- 1. Convert the following  $n^{\text{th}}$  order DEs to systems of first order.
  - (a) y'' + 3y' + 2y = 0
  - (b) y''' = 2y 3y'
  - (c) 2y'' + 6y' + y = 0
  - (d) y'' + 5y = 0
  - (e)  $y^{(v)} = y 3y' + ty'' + y''' 3y^{(iv)}$
- 2. Convert the following systems of first order to an equivalent second order DE, if possible.
  - (a)  $\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned}$ (b)  $\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= x_1 + x_2 \end{aligned}$ (c)  $\begin{aligned} x_1' &= x_1 \\ x_2' &= 3x_2 \end{aligned}$