Math 244 Sample Final A Solutions

1. Find values of k for which the IVP: $ty' - 4y = 0$, $y(0) = k$ has (i) No solution, (ii) An infinite number of solutions. Does this violate the Existence and Uniqueness Theorem (explain)?

   SOLUTION: Writing the DE as separable, we have $y' = \frac{4y}{t}$, so separating variables we have

$$\frac{1}{y}\,dy = \frac{4}{t}\,dt \quad \Rightarrow \quad \ln|y| = 4\ln|t| + C = \ln|t^4| + C$$

   Exponentiating both sides, $y = At^4$. Putting in the initial condition $t = 0$ forces $k = 0$ and A can be any real number.

   For (i), if $k \neq 0$ we have no solution. For (ii), if $k = 0$, we have an infinite number of solutions. For the E&U theorem, $f(t, y) = 4y/t$, and so f is not continuous at $t = 0$. Thus, the theorem does not apply to this DE.

2. Suppose you have a tank of brine containing 300 gallons of water with a concentration of 1/6 pounds of salt per gallon. There is brine pouring into the tank at a rate of 3 gallons per minute, and it contains 2 pounds of salt per gallon. The well-mixed solution leaves at 2 gallons per minute. (i) Write the initial value problem for the amount of salt in the tank at time t, and (ii) solve it.

   SOLUTION: Using $s(t)$ for salt in the tank at time t, and recall that ds/dt is in pounds (lbs) per minute,

$$\frac{ds}{dt} = 3\frac{\text{gal}}{\text{min}} \cdot 2\frac{\text{lbs}}{\text{gal}} - 2\frac{\text{gal}}{\text{min}} \cdot \frac{\text{s(t) lbs}}{(300+t)\text{ gal}}, \quad s(0) = \frac{300}{6} = 50 \text{ lbs}$$

   To solve this, we re-write it as a linear DE:

$$s' + \frac{2}{300+t}s = 6 \quad \Rightarrow \quad \mathrm{e}^{\int \frac{2}{300+t}\,dt} = (300+t)^2 \quad \Rightarrow$$

$$((300+t)^2 s)' = 6(300+t)^2 \quad \Rightarrow \quad (300+t)^2 s = 2(300+t)^3 + C \quad \Rightarrow \quad s(t) = 2(300+t) + \frac{C}{(300+t)^2}$$

   Solving for C, we get:

$$50 = 600 + \frac{C}{300^2} \quad \Rightarrow \quad C = (-550)(300^2)$$

   It's fine to leave C like this since we aren't using a calculator.

3. For the following, find the power series expansion for the general solution up to and including the t^4 term: $y' - 2y = \sin(t)$

   SOLUTION: Since this is first order, $y(0) = a_0$ will be arbitrary. Then:

$$y'(t) = 2y + \sin(t) \quad \Rightarrow \quad y'(0) = 2a_0 + 0 = 2a_0$$

$$y''(t) = 2y'(t) + \cos(t) \quad \Rightarrow \quad y''(0) = 2(2a_0) + 1 = 1 + 4a_0$$

$$y'''(t) = 2y''(t) - \sin(t) \quad \Rightarrow \quad y'''(0) = 2(1 + 4a_0) + 0 = 2 + 8a_0$$

$$y^(4)(t) = 2y^(3)(t) - \cos(t) \quad \Rightarrow \quad y^(4)(0) = 2(2 + 8a_0) - 1 = 3 + 16a_0$$

   Using the Maclaurin series formula: $a_n = y^{(n)}(0)/n!$, we have:

$$y(t) \approx a_0 + 2a_0 t + \frac{1 + 4a_0}{2!}t^2 + \frac{2 + 8a_0}{3!}t^3 + \frac{3 + 16a_0}{4!} + \cdots$$

4. Using the method of undetermined coefficients, give the form of the particular solution (do not solve) to:

$$y'' - 6y' + 9y = 6t^2 - 12te^{3t}$$

SOLUTION: We want to check the homogeneous part: $\lambda^2 - 6\lambda + 9 = 0$, so $\lambda = 3, 3$ and

$$y_h(t) = C_1 e^{3t} + C_2 t e^{3t}$$

We can further break up the particular part- Our guess using $6t^2$ would be:

$$y_{p_1}(t) = At^2 + Bt + C$$

and for $-12te^{3t}$, we'll need to multiply by t^2:

$$y_{p_2}(t) = t^2(Dt + E)e^{3t}$$

5. Classify the origin using the Poincaré Diagram and solve using eigenvectors/eigenvalues, then provide a sketch of the phase portrait:

$$\mathbf{Y}'(t) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \mathbf{Y}$$

SOLUTION: The trace is -2, the determinant is 5 and the discriminant is $4 - 4 \cdot 5 = -16$. Therefore, the origin is a spiral sink. For the eigenvalues/eigenvectors, we could use these values, but we'll start from scratch as a reminder:

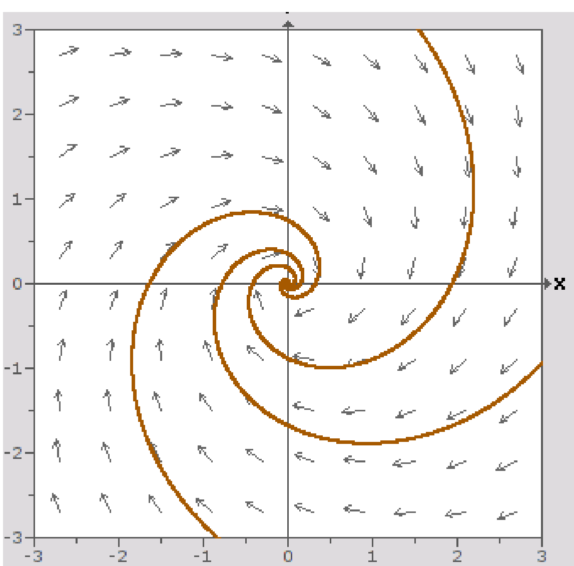
$$\begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda+1)^2 = -4 \quad \rightarrow \quad \lambda = -1 \pm 2i$$

Using $\lambda = -1 + 2i$, we get an eigenvector:

$$(-1 - (-1 + 2i))v_1 + 2v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 2i \end{bmatrix} \quad \text{or } \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

We need the real and imaginary parts of $e^{\lambda t}\mathbf{v}$, which are:

$$y(t) = C_1 e^{-t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$



6. Solve:

   (a) $y' = -\frac{y}{1+t} + t^2$
   SOLUTION: This equation is linear- Use an integrating factor:

$$y' + \frac{1}{1+t}y = t^2 \qquad \text{so that } e^{\int p(t)\,dt} = 1+t$$

   The solution is given by:

$$y(t) = \frac{\frac{1}{4}t^4 + \frac{1}{3}t^3 + C}{1+t}$$

   (b) $y' = y(3-2y)$
   SOLUTION: This equation is autonomous (actually, separable):

$$\int \frac{1}{y(3-2y)}\,dy = \int dt \quad \Rightarrow \quad \int \frac{1}{3}\frac{1}{y} + \frac{2}{3}\frac{1}{3-2y}\,dy = t + C$$

$$\frac{1}{3}\ln(y) - \frac{1}{3}\ln(3-2y) = t + C \quad \Rightarrow \quad \frac{y}{3-2y} = Ae^{3t} \quad \Rightarrow \quad y(t) = \frac{3Ae^{3t}}{1+2Ae^{3t}}$$

   (c) $t\frac{dy}{dt} - (1+t)y = ty^2$ First, use the substitution: $u = y^{-1}$ to get a DE in u.
   SOLUTION: Using the suggested substitution, we can write $y = 1/u$, so $y' = -u'/u^2$, so

$$\frac{-t}{u^2}u' - (1+t)\frac{1}{u} = \frac{t}{u^2}$$

   Multiply both sides by $-u^2/t$ to get the coefficient of u' equal to one, and we have our linear DE:

$$u' + \frac{1+t}{t}u = -1 \quad \Rightarrow \quad e^{\int p(t)\,dt} = te^t$$

   Now,

$$(te^t u)' = -te^t \quad \Rightarrow \quad te^t u = -te^t + e^t + C$$

   so

$$u(t) = -1 + \frac{1}{t} + \frac{Ce^{-t}}{t} \quad \Rightarrow \quad y(t) = \frac{t}{1 - t + Ce^{-t}}$$

7. Solve for the Laplace Transform, $Y(s)$, of the solution $y(t)$ (do not invert the transform):

$$y'' + 6y' + 5y = t - t^2 u_2(t), \quad y(0) = 1, y'(0) = 0$$

   SOLUTION: You might note that $\mathcal{L}(u_2(t)t^2)$ should be done on the side, since

$$f(t-2) = t^2 \quad \Rightarrow \quad f(t) = (t+2)^2 = t^2 + 4t + 4$$

   Therefore,

$$\mathcal{L}(u_2(t)t^2) = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

   The rest of the pieces are straightforward:

$$(s^2 + 6s + 5)Y(s) = \frac{1}{s^2} - 2e^{-2s}\frac{2s^2 + 2s + 1}{s^3} + s + 6$$

   so that

$$Y(s) = \frac{1}{s^2(s^2 + 6s + 5)} - 2e^{-2s}\frac{2s^2 + 2s + 1}{s^3(s^2 + 6s + 5)} + \frac{s+6}{s^2 + 6s + 5}$$

8. Compute $\mathcal{L}^{-1}\left(\dfrac{s}{s^2-10s+29}\right)$

SOLUTION: Completing the square in the denominator,

$$\frac{s}{s^2-10s+25+4}=\frac{s}{(s-5)^2+2^2}=\frac{s-5}{(s-5)^2+2^2}+\frac{5}{2}\frac{2}{(s-5)^2+2^2}$$

so that the inverse Laplace transform is given by

$$e^{5t}\cos(2t)+\frac{5}{2}e^{5t}\sin(2t)$$

9. Write the solution to the following DE in terms of $g(t)$: $y''+4y=g(t)$, $y(0)=3, y'(0)=-1$.

SOLUTION: We can do this via the Laplace transform:

$$s^2Y-3s+1+4Y=G(s)\quad\Rightarrow\quad(s^2+4)Y(s)=G(s)+3s-1\quad\Rightarrow\quad Y(s)=G(s)\frac{1}{s^2+4}+\frac{3s-1}{s^2+4}$$

For the second term,

$$\frac{3s-1}{s^2+4}=3\frac{s}{s^2+4}-\frac{1}{2}\frac{2}{s^2+4}$$

so that the inverse Laplace transform yields the following solution:

$$y(t)=3\cos(2t)-\frac{1}{2}\sin(2t)+g(t)*\frac{1}{2}\sin(2t)$$

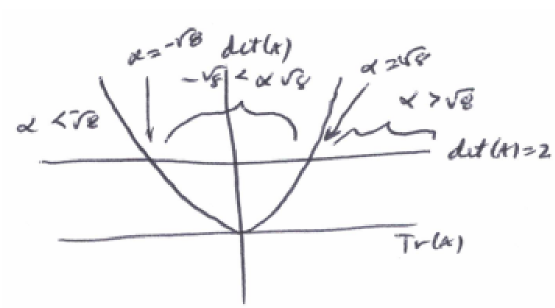
10. Given the system of equations below, describe (using the Poincaré Diagram) how the classification of the origin changes with α.

$$\mathbf{Y}'=\begin{bmatrix}\alpha & 1\\ -2 & 0\end{bmatrix}\mathbf{Y}$$

We see that $\mathrm{Tr}(A)=\alpha$, $\det(A)=2$, and $\Delta=\alpha^2-8$. Algebraically, we see that:

$\mathrm{Tr}(A)=\alpha$	$-$	$-$	$+$	$+$
$\det(A)=2$	$+$	$+$	$+$	$+$
$\Delta=\alpha^2-8$	$+$	$-$	$-$	$+$
	$\alpha<-\sqrt{8}$	$-\sqrt{8}<\alpha<0$	$0<\alpha<\sqrt{8}$	$\alpha>\sqrt{8}$
	Sink	Spiral Sink	Spiral Source	Source

We also have $\alpha=-\sqrt{8}$ and $\alpha=\sqrt{8}$ correspond to degenerate sink/source respectively, and if $\alpha=0$, we have a center.



11. Suppose that our mass-spring system is given by $y''+3y'+y=\cos(\omega t)$.

(a) Is there any value of ω that would give us resonance? No resonance. Beating? No beating.

(b) Find the value of ω that gives the maximum amplitude for the particular solution.
SOLUTION: When we complexify and solve using $y_p = Ae^{i\omega t}$, we get:

$$A = \frac{1}{(1-\omega^2) + 3\omega i} \quad \Rightarrow \quad |A| = \frac{1}{\sqrt{(1-\omega^2)^2 + 9\omega^2}}$$

If we differentiate, we find that there are no real values of ω that will make the derivative equal zero, so there is no maximum. I didn't mean to do that, but it is worth thinking about- If we had switched the damping coefficient and spring constant so that $y'' + y' + 3y = \cos(\omega t)$, there would have been a solution- In this case, there was too much damping!

12. The graph below is $y' = f(y)$.

(a) Locate and classify all equilibria.
Equilibria are at $y = 0, 1, 2, 3$.

(b) Provide a sketch of the direction field.
See sketch.

(c) Give one interval on which $y(t)$ is concave up.
See sketch.

(d) True or False? The solution $y(t)$ may be periodic.
False- y' does not depend on t, which would need to be true if y was periodic.