Math 244 Sample Final B

Show all your work! A table of Laplace transforms is provided.

1. Short Answer:
   (a) Suppose that $y' = \mathrm{e}^y + \frac{t^2}{\mathrm{e}^y}$. Substitute $u = \mathrm{e}^y$ to get a differential equation in u.
   (b) Use the method of undetermined coefficients to give the form of the particular solution(s) to the following (DO NOT solve for the coefficients!):
   $$y'' - 6y' + 9y = 6t^2 - 12t\mathrm{e}^{3t} + t\sin(2t)$$

2. Short Answer, Laplace Transforms:
   (a) Define the Laplace transform for $f(t)$: $F(s) =$
   (b) Compute the inverse Laplace transform of $\dfrac{s}{s^2 - 10s + 29}$
   (c) Compute the Laplace transform of the following (leave your answer as $Y(s) = ...$)
   $$y'' + 6y' = t^2 u_2(t) \quad y(0) = 1 \quad y'(0) = 0$$

3. Given $y' = y - y^2$, answer the following without solving the DE:
   (a) If $y(0) = 2$, what is the limit $t \to \infty$ of $y(t)$?
   (b) Find where $y(t)$ is increasing, decreasing.
   (c) Find where $y(t)$ is concave up, concave down.

4. Suppose you have a tank of brine containing 300 gallons of water with a concentration of 1/6 pounds of salt per gallon. There is brine pouring into the tank at a rate of 3 gallons per minute, and it contains 2 pounds of salt per gallon. The well-mixed solution leaves at 2 gallons per minute. (i) Write the initial value problem for the amount of salt in the tank at time t, and (ii) solve it.

5. Classify the origin using the Poincaré Diagram and solve the system using eigenvectors/eigenvalues:
   $$\mathbf{Y}'(t) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \mathbf{Y}$$

6. Solve the same system as in Problem (5), but solve it by first writing an equivalent second order differential equation (be sure to solve for both $y_1(t)$ and $y_2(t)$).

7. Find the series expansion for the following, up to and including the t^4 term, where
   $$y'' - ty' + y = 0 \qquad y(1) = 1 \quad y'(1) = 2$$

8. Solve:
   (a) $y' = y(1 - y)$
   (b) $ty' + ty = 1 - y$
   (c) $y'' + 2y' = t^2$

9. Let
   $$\begin{aligned} x' &= 2x - x^2 - xy \\ y' &= y - xy \end{aligned}$$
   (a) If $x(t)$ and $y(t)$ are populations of animals, is this differential equation a version of "predator-prey" or "competing species"? Explain- In particular, discuss the assumptions on how the populations increase/decrease in the absence of the other.
   (b) Find and classify all equilibrium solutions.
   (c) Using your previous answers, if $x_0 = \frac{1}{2}$, and $y_0 = \frac{1}{8}$, where does $(x(t), y(t))$ go as $t \to \infty$?