Math 244 Sample Final B Solutions

1. Short Answer:

   (a) Suppose that $y' = e^y + \frac{t^2}{e^y}$. Substitute $u = e^y$ to get a differential equation in $u$.
   SOLUTION: With the given substitution, $\frac{du}{dt} = e^y \frac{dy}{dt} = u\frac{dy}{dt}$, so:

$$\frac{u'}{u} = u + \frac{t^2}{u} \quad \Rightarrow \quad u' = u^2 + t^2$$

   (b) Use the method of undetermined coefficients to give the form of the particular solution(s) to the following (DO NOT solve for the coefficients!):

$$y'' - 6y' + 9y = 6t^2 - 12te^{3t} + t\sin(2t)$$

   SOLUTION: Split the guesses into 3, note that $y_h = e^{3t}(C_1 + C_2 t)$:

$$y_{p_1} = At^2 + Bt + C \qquad y_{p_2} = t^2(At + B)e^{3t} \qquad y_{p_3} = (At + B)\cos(2t) + (Ct + D)\sin(2t)$$

2. Short Answer, Laplace Transforms:

   (a) Define the Laplace transform for $f(t)$: $F(s) = \displaystyle\int_0^\infty e^{-st} f(t)\,dt$

   (b) Compute the inverse Laplace transform of $\dfrac{s}{s^2 - 10s + 29}$

$$\frac{s}{(s-5)^2 + 2^2} = \frac{s-5}{(s-5)^2 + 2^2} + \frac{5}{2}\frac{2}{(s-5)^2 + 2^2} \quad \to \quad e^{5t}\cos(2t) + \frac{5}{2}e^{5t}\sin(2t)$$

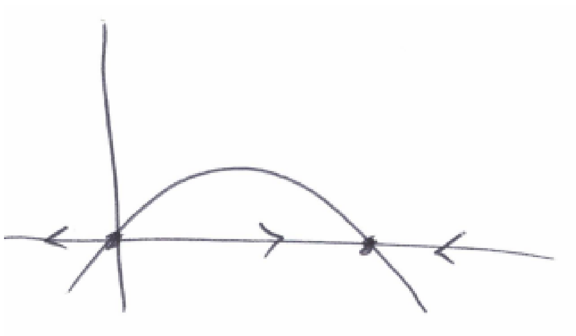
   (c) Compute the Laplace transform of the following (leave your answer as $Y(s) = ...$)

$$y'' + 6y' = t^2 u_2(t) \quad y(0) = 1 \quad y'(0) = 0$$

   First, note that the Laplace transform of the right side will have you take $f(t-2) = t^2$, so that $f(t) = (t+2)^2 = t^2 + 4t + 4$. Therefore,

$$Y(s) = \frac{e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)}{s^2 + 6s} + \frac{s+6}{s^2 + 6s}$$

3. Given $y' = y - y^2$, answer the following without solving the DE:
   SOLUTION: See the graph for the details.



   (a) If $y(0) = 2$, what is the limit $t \to \infty$ of $y(t)$? SOLN: $y(t) \to 1$

(b) Find where $y(t)$ is increasing, decreasing.

SOLN: The function $y(t)$ is increasing if $0 < y < 1$, decreasing if $y < 0$ or $y > 1$.

(c) Find where $y(t)$ is concave up, concave down.

SOLN: Recall that $\frac{d^2y}{dt^2} = \frac{df}{dy}\frac{dy}{dt}$, so we have:

If $y < 0$, then $y$ is concave down. If $0 < y < 1/2$, the $y$ is concave up, and if $1/2 < y < 1$, then $y$ is concave down. Finally, if $y > 1$, then $y$ is concave up.

4. Suppose you have a tank of brine containing 300 gallons of water with a concentration of 1/6 pounds of salt per gallon. There is brine pouring into the tank at a rate of 3 gallons per minute, and it contains 2 pounds of salt per gallon. The well-mixed solution leaves at 2 gallons per minute. (i) Write the initial value problem for the amount of salt in the tank at time $t$, and (ii) solve it.

SOLUTION: Sorry- This is the same problem as Version A. Try another one from the text, like #27, p. 135 (Section 1.9).

5. Classify the origin using the Poincaré Diagram and solve the system using eigenvectors/eigenvalues:

$$\mathbf{Y}'(t) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \mathbf{Y}$$

SOLUTION: Sorry- This is the same problem as Version A. Try a couple from #27-32 on page 380.

6. Solve the same system as in Problem (5), but solve it by first writing an equivalent second order differential equation (be sure to solve for both $y_1(t)$ and $y_2(t)$).

SOLUTION: First, we'll re-write it in system form, then we'll convert it to second order:

$$\begin{array}{ll} y_1' & = -y_1 + 2y_2 \\ y_2' & = -2y_1 - y_2 \end{array} \quad \Rightarrow \quad \begin{array}{c} y_2 = (1/2)(y_1' + y_1) \\ (1/2)(y_1'' + y_1') = -2y_1 - (1/2)(y_1' + y_1) \end{array}$$

Multiplying the DE by 2, we get $y_1'' + 2y_1' + 5y_1 = 0$, from which we get the characteristic equation $\lambda^2 + 2\lambda + 5 = 0$, or

$$\lambda = -1 \pm 2i$$

so that

$$y_1(t) = \mathrm{e}^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$$

To find $y_2(t)$, take $\frac{1}{2}(y_1' + y_1)$- In this case, that's a bit of algebraic work- You might try it for practice if you have time.

7. Find the series expansion for the following, up to and including the $t^4$ term, where

$$y'' - ty' + y = 0 \qquad y(1) = 1 \quad y'(1) = 2$$

SOLUTION: First note that we are evaluating at $t = 1$ instead of $t = 0$. This means that our power series solution is of the form:

$$y(t) = a_0 + a_1(t-1) + a_2(t-1)^2 + a_3(t-1)^3 + \cdots$$

And because we have a second order equation, we have $y(1) = a_0$ and $y'(1) = a_1$ as arbitrary constants (so we should be able to write all other constants in terms of these two).

Now proceed as usual to compute the derivatives of $y$ at $t = 1$:

$$y'' = ty' - y \quad \Rightarrow \quad y''(1) = y'(1) - y(1) = a_1 - a_0$$

$$y''' = y' + ty'' - y' = ty'' \quad \Rightarrow \quad y'''(1) = a_1 - a_0$$

$$y^{(4)} = y'' + ty''' \quad \Rightarrow \quad y^{(4)}(1) = (a_1 - a_0) + (a_1 - a_0) = 2(a_1 - a_0)$$

Using the fact that $a_n = y^{(n)}(1)/n!$, we can write the series as:

$$y(t) \approx a_0 + a_1(t-1) + \frac{a_1 - a_0}{2}(t-1)^2 + \frac{a_1 - a_0}{3!}(t-1)^3 + \frac{2(a_1 - a_0)}{4!}(t-1)^4 + \cdots$$

8. Solve:

   (a) $y' = y(1-y)$

   SOLUTION: This is separable (as well as autonomous):

$$\int \frac{1}{y(1-y)}\, dy = \int dt \quad \Rightarrow \quad \ln(y) - \ln(1-y) = t + C \quad \Rightarrow \quad \frac{y}{1-y} = A\mathrm{e}^t \quad \Rightarrow \quad y(t) = \frac{A\mathrm{e}^t}{1 + A\mathrm{e}^t}$$

   (b) $ty' + ty = 1 - y$

   SOLUTION: Linear with integrating factor. Rewrite so that

$$y' + (1 + \frac{1}{t})y = \frac{1}{t}$$

   The integrating factor is then $t\mathrm{e}^t$:

$$(t\mathrm{e}^t y)' = \mathrm{e}^t \quad \Rightarrow \quad y(t) = \frac{1 + C\mathrm{e}^{-t}}{t}$$

   (c) $y'' + 2y' = t^2$

   SOLUTION: Second order with a forcing function. Solving the homogeneous equation $y'' + 2y' = 0$, we have

$$\lambda^2 + 2\lambda = 0 \quad \Rightarrow \quad \lambda = 0, \lambda = -2 \quad \Rightarrow \quad y_h(t) = C_1 + C_2\mathrm{e}^{-2t}$$

   For the particular solution, $y_p = At^2 + Bt + C$, but since a constant is part of the homogeneous solution, multiply by $t$ so that $y_p = At^3 + Bt^2 + Ct$. Put that into the DE, we should find that

$$y_p(t) = \frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{1}{4}t$$

   For the full solution, we have

$$y(t) = C_2\mathrm{e}^{-t} + C_1 + \frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{1}{4}t$$

9. Let

$$\begin{aligned} x' &= \; 2x - x^2 - xy \\ y' &= \; y - xy \end{aligned}$$

   (a) If $x(t)$ and $y(t)$ are populations of animals, is this differential equation a version of "predator-prey" or "competing species"? Explain- In particular, discuss the assumptions on how the populations increase/decrease in the absence of the other.

   SOLUTION: This is an examples of competing species- That is, when the populations interact, both rates of change are negative. Further, in the absence of $y$, the rate of change of $x$ is modeled after a logistic equation. In the absense of $x$, $y$ is modeled after exponential growth.

   (b) Find and classify all equilibrium solutions.

   SOLUTION: From the second equation, $y = 0$ or $x = 1$. If $y = 0$ for the second equation, then in the first we have $x(2-x) = 0$. Therefore, we have $(0,0)$ and $(2,0)$ for equilibria.

If $x = 1$ in the second equation, in the first we have $1 - y = 0$, so $y = 1$. Thus, $(1, 1)$ is the third solution.

To classify equilibria, we need the Jacobian matrix:

$$J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2 - 2x - y & -x \\ -y & 1 - x \end{bmatrix}$$

At the three points: $(0, 0)$, $(2, 0)$ and $(1, 1)$ respectively:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$$

Our analysis should suggest a SOURCE, a SADDLE, and a SINK (respectively).

(c) Using your previous answers, if $x_0 = \frac{1}{2}$, and $y_0 = \frac{1}{8}$, where does $(x(t), y(t))$ go as $t \to \infty$?

SOLUTION: Consider a brief sketch of the direction field (see below). It's clear that the point should be attracted to the sink at $(2, 0)$ and not to the saddle at $(1, 1)$. Therefore, we conclude that, as time gets larger and larger, $x(t) \to 2$ and $y(t)$ dies off.