Sample Final Exam C

1. Let $y'' - ty' - y = 0$. If $y(1) = 1$ and $y'(1) = 1$, write the power series for y up to and including the fourth order term.

2. Short Answer:

   (a) Convert $-1 + i$ to polar form.

   (b) Find the Laplace transform of $\delta_2(t)t^2$.

   (c) Find $t * \mathrm{e}^t$ using the definition of the convolution, and verify your computation using the Laplace transform.

   (d) Convert the third order DE to a system of first order: $y''' - t^2y'' + 2y = 0$

   (e) Convert the following system to an equivalent second order DE: $\begin{aligned} x_1' &= 3x_1 + x_2 \\ x_2' &= x_1 + 3x_2 \end{aligned}$

3. Show, with the substitution $v = y/x$, that the following DE become linear (in v): $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$

4. Consider the IVP: $y' = \dfrac{t}{y - yt^2}$, $y(0) = 4$.

   - What does the Existence and Uniqueness theorem say about the solution(s) to the IVP (be specific in what you're computing).
   - Solve the IVP. You may leave your answer in implicit form.

5. Compute the inverse Laplace transform of $\dfrac{\mathrm{e}^{2s}(s+5)}{s^2 + 2s + 3}$

6. Solve:

   (a) $ty' + 2y = \sin(t)$ (you may assume $t > 0$).

   (b) $y' + y^2 \sin(x) = 0$

   (c) $y'' - 5y' = t^2$

   (d) $\mathbf{Y}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \mathbf{Y}$

7. Solve $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$, where $g(t) = \begin{cases} t/2 & \text{if } 0 \le t \le 6 \\ 3 & \text{if } t \ge 6 \end{cases}$

8. The air in a small room, 20 ft by 5 ft by 10 ft is 3% carbon monoxide. Starting at $t = 0$, air containing 1% carbon monoxide is blown into the room at a rate of 100 ft^3 per hour, and the well mixed air flows out through a vent at the same rate.

   (a) Write the IVP modeling the amount of carbon monoxide in the room at time t.

   (b) Give a graphical analysis of the solution. In particular, what happens in the room over the long term.

   (c) Solve the IVP.

9. Consider $y'' + \frac{1}{10}y' + y = \cos(\omega t)$.

   - Does the solution have *beating* or *resonance*? Give a short reason.
   - Consider the homogeneous differential equation. Is it overdamped, underdamped, or critically damped?

- For the forced system with $\omega = 1$, find the particular solution.
- Going back to the general ω, find the amplitude of the particular solution (in terms of ω).

10. Let

$$\begin{aligned} x' &= 1 - x^2 - y^2 \\ y' &= 2xy \end{aligned}$$

(a) Draw the nullclines and sketch the direction field along the nullclines.

(b) Find and classify all equilibria (Poincaré)