Sample Final Exam C

1. Let $y'' - ty' - y = 0$. If $y(1) = 1$ and $y'(1) = 1$, write the power series for y up to and including the fourth order term.

   SOLUTION: This ended up being almost identical to the one on Sheet B (oops!). It does have different starting values, so we will have different constants:

$$y'' = ty' + y \quad \Rightarrow \quad y''(1) = y'(1) + y(1) = 2$$

$$y''' = 2y' + ty'' \quad \Rightarrow \quad y'''(1) = 2(1) + (1)(2) = 4$$

$$y^{(4)} = 3y'' + ty''' \quad \Rightarrow \quad y^{(4)}(1) = 3(2) + (1)(4) = 10$$

   Therefore,

$$y(t) \approx 1 + (t-1) + \frac{2}{2!}(t-1)^2 + \frac{4}{3!}(t-1)^3 + \frac{10}{4!}(t-1)^4 + \cdots$$

2. Short Answer:

   (a) Convert $-1 + i$ to polar form.
   SOLUTION: Writing as $re^{i\theta}$, we have $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

   (b) Find the Laplace transform of $\delta_2(t)t^2$.
   SOLUTION: Recall that $\delta_c(t)f(t)$ is the same thing as $\delta_c(t)f(c)$, since the delta function is zero everywhere except at c. Therefore, we can rewrite this as a constant times the delta function, and then apply Laplace:

$$\delta_2(t)t^2 = 4\delta_2(t) \quad \rightarrow \quad 4e^{-2s}$$

   (c) Find $t * e^t$ using the definition of the convolution, and verify your computation using the Laplace transform.
   SOLUTION: Using the definition of the convolution,

$$t * e^t = \int_0^t (t-u)e^u \, du = t\int_0^t e^u \, du - \int_0^t ue^u \, du$$

   The second integral needs to be integrated by parts- Using a table makes it quick:

$$\begin{array}{ccc} + & u & e^u \\ - & 1 & e^u \\ + & 0 & e^u \end{array} \quad \Rightarrow \quad (ue^u - e^u|_0^t = te^t - e^t + 1$$

   Including that first integral, simplify to get $-t - 1 + e^t$.
   Notice that we could have integrated the other way as well:

$$\int_0^t e^{t-u} u \, du = e^t \int_0^t u \, e^{-u} \, du$$

   Working through this integral, we get the same answer.
   Finally, we show we get the same answer if we use the Laplace transform.

$$\mathcal{L}(t * e^t) = \frac{1}{s^2}\frac{1}{s-1} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} = \frac{-1}{s} + \frac{-1}{s^2} + \frac{1}{s-1}$$

   Now, the inverse Laplace is $-1 - t + e^t$, as we got before.

(d) Convert the third order DE to a system of first order: $y''' - t^2y'' + 2y = 0$

SOLUTION: Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$. We write the system of equations in x_i- Remember to include them all, and remember to put them in order!

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -2x_1 + t^2x_3 \end{aligned}$$

(e) Convert the following system to an equivalent second order DE: $\begin{aligned} x_1' &= 3x_1 + x_2 \\ x_2' &= x_1 + 3x_2 \end{aligned}$

SOLUTION: Use one equation to get one variable in terms of the other, then use the other equation to remove it. For example, here we'll use the first equation to solve for x_2:

$$x_2 = x_1' - 3x_1$$

Then we substitute this into the second equation:

$$(x_1' - 3x_1)' = x_1 + 3(x_1' - 3x_1) \quad \Rightarrow \quad x_1'' - 6x_1' + 8x_1 = 0$$

3. Show, with the substitution $v = y/x$, that the following DE become linear (in v): $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$

SOLUTION: There was an error here- This becomes **separable**, not linear. Continuing, we see that $y = vx$, so we substitute $y' = v'x + v$. Direct substitution:

$$v'x + v = \frac{4vx - 3x}{2x - vx} = \frac{4v - 3}{2 - v} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{4v - 3}{2 - v} - v = \frac{v^2 + 2v - 3}{2 - v}$$

So now its separable:

$$\frac{2 - v}{v^2 + 2v - 3}\, dv = \frac{1}{x}\, dx$$

4. Consider the IVP: $y' = \dfrac{t}{y - yt^2}$, $y(0) = 4$.

- What does the Existence and Uniqueness theorem say about the solution(s) to the IVP (be specific in what you're computing).
  SOLUTION: Here, $f(t, y) = \frac{1}{y} \cdot \frac{t}{1-t^2}$, which is continuous as long as we stay away from $y = 0$ and $t = \pm 1$. Since we start at $t = 0$ and $y = 4$, a solution should exist. Also, $f_y(t, y) = -\frac{1}{y^2} \cdot \frac{t}{1-t^2}$, which does not add any more discontinuities.
  Conclusion: For our IVP, we expect a unique solution.
- Solve the IVP. You may leave your answer in implicit form.
  SOLUTION: The equation is separable. Separate variables and integrate both sides to get

$$\frac{1}{2}y^2 = -\frac{1}{2}\ln(1 - t^2) + C$$

5. Compute the inverse Laplace transform of $\dfrac{e^{2s}(s + 5)}{s^2 + 2s + 3}$

TYPO: The exponential should be e^{-2s}.

SOLUTION: Think of this as $e^{-2s}F(s)$, so when we invert it, the answer will be $u_2(t)f(t-2)$. Focusing on just $F(s)$ then:

$$\frac{s + 5}{s^2 + 2s + 3} = \frac{s + 1}{(s + 1)^2 + 2} + \frac{4}{\sqrt{2}}\frac{\sqrt{2}}{(s + 1)^2 + 2}$$

Therefore,

$$f(t) = \mathrm{e}^{-t}\cos(\sqrt{2}\,t) + \frac{4}{\sqrt{2}}\mathrm{e}^{-t}\sin(\sqrt{2}\,t)$$

where the solution will be $u_2(t)f(t-2)$.

6. Solve:

   (a) $ty' + 2y = \sin(t)$ (you may assume $t > 0$).
   SOLUTION: Linear DE. Put the equation in standard form

   $$y' + \frac{2}{t}y = \frac{\sin(t)}{t}$$

   so that the integrating factor is $\mathrm{e}^{\int p(t)\,dt} = t^2$. Continuing, we note that the integral of $t\sin(t)$ is found using integration by parts:

   $$(t^2y)' = t\sin(t) \quad \Rightarrow \quad t^2y = -t\cos(t) + \sin(t) + C \quad \Rightarrow \quad y(t) = \frac{-t\cos(t) + \sin(t) + C}{t^2}$$

   (b) $y' + y^2\sin(x) = 0$
   SOLUTION: Nonlinear, but separable.

   $$\int \frac{-1}{y^2}\,dy = \int \sin(x)\,dx \quad \Rightarrow \quad \frac{1}{y} = -\cos(x) + C \quad \Rightarrow \quad y(x) = \frac{1}{C - \cos(x)}$$

   (c) $y'' - 5y' = t^2$
   SOLUTION: This is almost the same as in Version B (Sorry!), but the technique is worth repeating- Solve for the homogeneous part, then the particular part using Method of Undetermined Coefficients. It is possible to use Laplace, but the algebra does get a bit messy. For the homogeneous part,

   $$\lambda(\lambda - 5) = 0 \quad y_h(t) = C_1 + C_2\mathrm{e}^{5t}$$

   The particular part: $y_p = (At^2 + Bt + C)t$. Substituting and solving for the coefficients, we get

   $$y(t) = C_1 + C_2\mathrm{e}^{5t} - \frac{1}{15}t^3 - \frac{1}{25}t^2 - \frac{2}{125}t$$

   (d) $\mathbf{Y}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}\mathbf{Y}$
   SOLUTION: We should find that the solution is given by

   $$Y(t) = C_1\mathrm{e}^{2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2\mathrm{e}^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7. Solve $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$, where $g(t) = \begin{cases} t/2 & \text{if } 0 \le t \le 6 \\ 3 & \text{if } t \ge 6 \end{cases}$

   SOLUTION: We first rewrite $g(t)$ in terms of Heaviside functions:

   $$g(t) = (1 - u_6(t))(t/2) + 3u_6(t) = \frac{t}{2} + u_6(t)\left(\frac{-(t-6)}{2}\right)$$

   Now we can take the Laplace transform of both sides:

   $$Y(s) = \frac{1}{2s^2(s^2+1)} - \frac{\mathrm{e}^{-6s}}{2s^2(s^2+1)} + \frac{1}{s^2+1}$$

We note that, for partial fractions, if we label the following $F(s)$ and invert:

$$F(s) = \frac{1}{2s^2(s^2+1)} = \frac{1}{2}\frac{1}{s^2} - \frac{1}{2}\frac{1}{s^2+1} \quad \Rightarrow \quad f(t) = \frac{1}{2}t - \frac{1}{2}\sin(t)$$

Now,

$$y(t) = f(t) - u_6(t)f(t-6) + \sin(t)$$

8. The air in a small room, 20 ft by 5 ft by 10 ft is 3% carbon monoxide. Starting at $t = 0$, air containing 1% carbon monoxide is blown into the room at a rate of 100 ft^3 per hour, and the well mixed air flows out through a vent at the same rate.

   (a) Write the IVP modeling the amount of carbon monoxide in the room at time t.
   SOLUTION: We'll use cubic feet for the CO. Then, at the initial time, there is 3% of 1000, or 30 ft^3 of CO. Therefore, let $M(t)$ be the cubic feet of CO at time t. We note that the rate of change will then have units ft^3/hr.

   $$\frac{dM}{dt} = \frac{1 \text{ ft}^3}{100 \text{ ft}^3} \cdot \frac{100 \text{ ft}^3}{1 \text{ hour}} - \frac{M \text{ ft}^3}{1000 \text{ ft}^3} \cdot \frac{100 \text{ ft}^3}{1 \text{ hour}} -$$

   Simplifying, we get:

   $$M' = 1 - \frac{1}{10}M, \qquad M(0) = 30$$

   (b) Give a graphical analysis of the solution. In particular, what happens in the room over the long term.
   SOLUTION: This is a line in the (M', M) plane with equilibrium solution at $M = 10$, which is a sink. Therefore, $M(t) \to 10$ as $t \to \infty$ (which is 1%).

   (c) Solve the IVP.
   SOLUTION: The DE is linear or separable. Probably easiest to solve as a linear DE:

   $$M(t) = 10 + 20\mathrm{e}^{-t/10}$$

9. Consider $y'' + \frac{1}{10}y' + y = \cos(\omega t)$.

   - Does the solution have *beating* or *resonance*? Give a short reason.
     SOLUTION: No beating or resonance, since the damping term is present.
   - Consider the homogeneous differential equation. Is it overdamped, underdamped, or critically damped?
     SOLUTION: $b^2 - 4ac < 0$, so we have complex roots to the characteristic equation. (UNDERDAMPED)
   - For the forced system with $\omega = 1$, find the particular solution.
     SOLUTION: With the ansatz $y_p(t) = A\mathrm{e}^{it}$, we find that $A = -10i$. Therefore, our solution is the real part of $A\mathrm{e}^{-it}$, or

     $$y_p(t) = 10\sin(t)$$

   - Going back to the general ω, find the amplitude of the particular solution (in terms of ω).
     SOLUTION: With a general $A\mathrm{e}^{i\omega t}$, we find that

     $$A = \frac{1}{(1-\omega^2) + \frac{1}{10}\omega i}$$

     so that the amplitude of the solution is $|A|$, or

     $$\frac{1}{\sqrt{(1-\omega^2)^2 + \frac{\omega^2}{100}}}$$

10. Let

$$\begin{aligned} x' &= 1 - x^2 - y^2 \\ y' &= 2xy \end{aligned}$$

(a) Draw the nullclines and sketch the direction field along the nullclines.

SOLUTIONS: The nullclines are the unit circle, $x = 0$ and $y = 0$. From this, we see that the equilibria are $(\pm 1, 0)$ and $(0, \pm 1)$.

For the sketch, for the circle in the upper left quadrant, arrows are pointing upward ($xy > 0$). They also point upward in Quadrant III, since both x, y are negative. In the other two quadrants, arrows along the circle point downward.

For $x = 0$ (the y-axis), inside the circle arrows point to the right, and outside the circle, to the left.

For $y = 0$ (the x-axis), inside the circle arrows point to the right, and outside the circle to the left.

(You can see these in the direction field below).

(b) Find and classify all equilibria (Poincaré)

You should find that $(1, 0)$ and $(-1, 0)$ are SADDLES and $(0, \pm 1)$ are both CENTERS (which we actually see in the direction field!)