Solutions: More Practice with Modeling

1. A mass weighing 4 lb stretches a spring 2 in. Suppose that the mass is displaced an additional 6 in the positive direction and released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/s. Formulate the IVP that governs the motion of the mass. (Hint: All units should be consistent. When working with US units, use pounds, feet and seconds.)

   SOLUTION: Remember to change units to pounds, feet and second. The first piece of information is given to find the spring constant. Use the relationship: $mg - kL = 0$, and the fact that pounds is already mg to get:

   $$4 - k\frac{1}{6} = 0 \quad \Rightarrow \quad k = 24$$

   Next, we need the damping constant, γ. We know that damping is proportional to the velocity, and the value of γ is non-negative, so we take the signs to be the appropriate values:

   $$F_{\text{damping}} = -\gamma y'(t) \quad \Rightarrow \quad 6 = \gamma 3 \quad \Rightarrow \quad \gamma = 2$$

   For the mass, remember that "pounds" is mg, and $g = 32$ (I would give you this value on an exam/quiz). For the mass,

   $$mg = 4 \quad \Rightarrow \quad m = \frac{4}{g} = \frac{4}{32} = \frac{1}{8}$$

   To finish, we have an initial position of 1/2 feet and zero initial velocity:

   $$\frac{1}{8}y'' + 2y' + 24y = 0 \qquad y(0) = \frac{1}{2} \quad y'(0) = 0$$

2. Suppose we consider a mass-spring system with no damping (the damping constant is then 0), so that the differential equation expressing the motion of the mass can be modeled as

   $$mu'' + ku = 0$$

   Find value(s) of β so that $A\cos(\beta t)$ and $B\sin(\beta t)$ are each solutions to the homogeneous equation (for arbitrary values of A, B).

   SOLUTION: We subsitute $u = A\cos(\beta t) + B\sin(\beta t)$ into the differential equation:

   $$m(-\beta^2 A\cos(\beta t) - \beta^2 B\sin(\beta t) + k(A\cos(\beta t) + B\sin(\beta t)) = 0$$

   We need the coefficient in front on $\cos(\beta t)$ to be zero:

   $$-m\beta^2 A + kA = 0$$

   and the coefficient in front of $\sin(\beta t)$ to be zero:

   $$-m\beta^2 B + kB = 0$$

   Take either equation to see that $\beta = \sqrt{\frac{k}{m}}$ (we'll only take the positive root).

3. A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in and then released, and if there is no damping, determine the IVP that governs the motion of the mass. (You might re-read the hint for (1))

   SOLUTION (remember units): Given the first line, we can find the value of the spring constant.
$$mg - kL = 0 \quad \Rightarrow \quad 2 - k\frac{1}{2} = 0 \quad \Rightarrow \quad k = 4$$
   For the mass,
$$mg = 2 \quad \Rightarrow \quad m = \frac{2}{g} = \frac{2}{32} = \frac{1}{16}$$
   Now we have the IVP:
$$\frac{1}{16}y'' + 4y = 0 \qquad y(0) = \frac{1}{4}, \quad y'(0) = 0$$

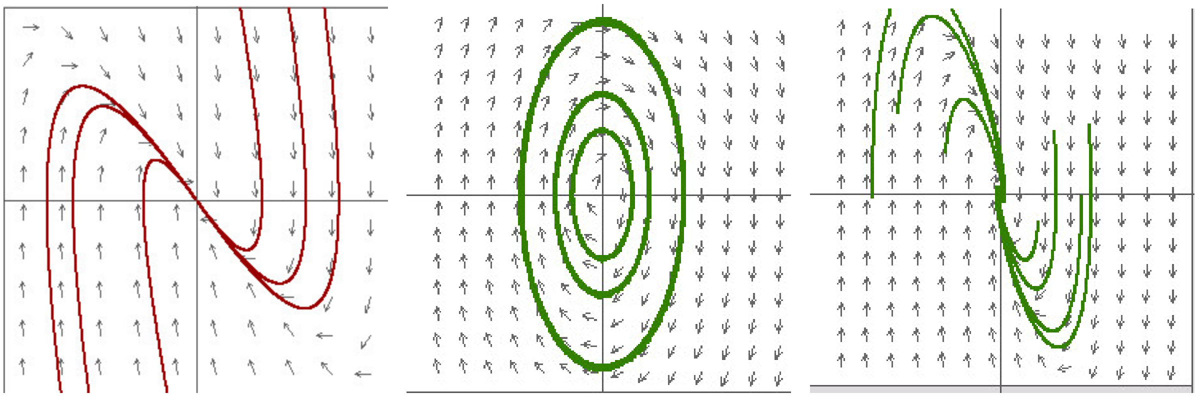
4. A mass of 20 grams stretches a spring 5 cm. Suppose tha the mass is attached to a viscous damper with a constant damping constant of 400 dyn-s/cm (note: a dyne is a unit of force using centimeters-grams- seconds for units). If the mass is pulled down an additional 2 cm then released, find the IVP that governs the motion of the mass.

   SOLUTION: The additional note tells us we can stick with the given units, but beware of g, which is normally 9.8 m/s^2, now it would be 980 cm/s^2. Given that,
$$mg - kL = 0 \quad \Rightarrow \quad (20)(980) - k(5) = 0 \quad \Rightarrow \quad k = 3920$$
   The damping constant is given as $\gamma = 400$, so now we put the equation together:
$$20y'' + 400y' + 3920y = 0 \quad y(0) = 2, \quad y'(0) = 0$$

5. For exercises 1, 3 and 4 above, convert the mass-spring second order DE into a system of first order equations, then analyze the solutions using the HPGSystemSolver software (in DETools).

   SOLUTION: When converting to a linear system $Y' = AY$, where A is:

$$(1) \begin{bmatrix} 0 & 1 \\ -192 & -16 \end{bmatrix} \qquad (3) \begin{bmatrix} 0 & 1 \\ -64 & 0 \end{bmatrix} \qquad (4) \begin{bmatrix} 0 & 1 \\ -196 & -20 \end{bmatrix}$$



6. A mass weighing 8 lbs stretches a spring $\frac{3}{2}$ in. The mass is attached to a damper with coefficient γ. Using the ansatz to get the characteristic equation for the second order linear, homogeneous, DE, find the value of γ at which we get one single real solution (to the characteristic equation).

   SOLUTION: As usual, the first line is used for both the spring constant k and the value of m:

$$mg - kL = 0 \quad \Rightarrow \quad 8 - k\frac{1}{8} = 0 \quad \Rightarrow \quad k = 64$$

   We also have:

$$8 = mg \quad \Rightarrow \quad m = \frac{8}{32} = \frac{1}{4}$$

   The DE is now:

$$\frac{1}{4}y'' + \gamma y' + 64y = 0 \quad \Rightarrow \quad y'' + 4\gamma y' + 256 = 0$$

   The ansatz is $y = \mathrm{e}^{st}$. Substituting this into the DE, we get

$$s^2 + 4\gamma s + 256 = 0$$

   We can use the quadratic formula to solve for s, but note that we want only ONE real solution- That happens when the discriminant is 0:

$$b^2 - 4ac = 0 \quad \Rightarrow \quad 16\gamma^2 - 4(256) = 0 \quad \Rightarrow \quad \gamma^2 = 64$$

   so that $\gamma = 8$ (only the positive one).