Review Questions, Exam 2

1. A weight of 4 lbs is attached to the end of a spring that is then stretched 3 inches. The spring is then attached to a dashpot that provides 6 lbs of resistance for each ft/s of velocity. If the spring is pushed up 6 inches and released, give the second order linear differential equation that models the position of the mass.

   SOLUTION: Recall that $4 = mg$, so for the spring constant, we take $mg - kL = 0$ (Remember to convert L to feet!):

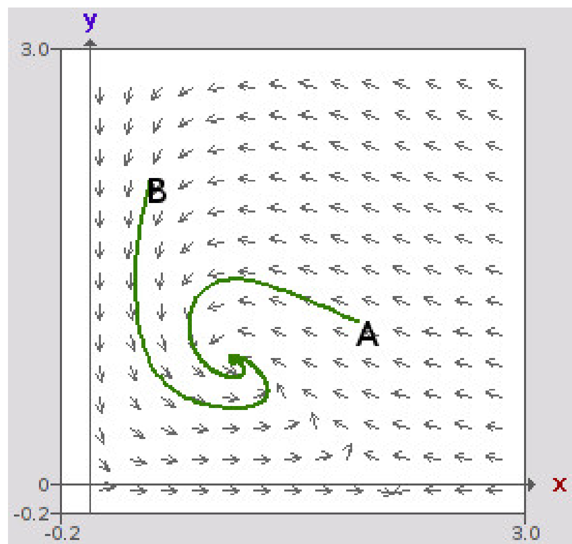
   $$4 - \frac{k}{4} = 0 \quad \Rightarrow \quad k = 16$$

   We're given the damping coefficient of 6, and $mg = 4$, so $m = 4/g = 4/32 = 1/8$:

   $$\frac{1}{8}y'' + 6y' + 16y = 0 \qquad y(0) = -\frac{1}{2}, \quad y'(0) = 0$$

2. Consider the system below, with two solution curves shown in the phase plane.

   $$\frac{dx}{dt} = x\left(2 - x - \frac{6}{5}y\right)$$
   $$\frac{dy}{dt} = y\,(-1 + x)$$



   (a) Find the equilibrium solutions.
   SOLUTION: From the first equation, we have

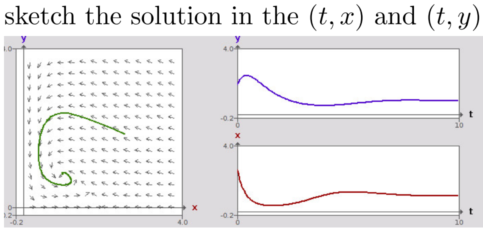
   $$x = 0 \qquad \text{or} \qquad x = 2 - \frac{6}{5}y$$

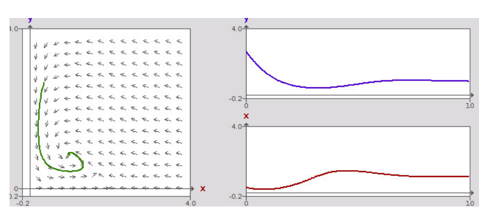
   If $x = 0$ from Equation 1, then Equation 2 becomes $y = 0$, so $(0, 0)$ is one equilibrium solution. If $x = 2 - \frac{6}{5}y$ from Equation 1, then Equation 2 becomes:

   $$y\left(-1 + \left(2 - \frac{6}{5}y\right)\right) \quad \Rightarrow \quad y = 0 \text{ or } y = \frac{5}{6}$$

   This gives $(2, 0)$ and $(1, 5/6)$ as the second and last equilibria.

   (b) Given the phase plane from `HPGSystemSolver` with starting points A and B as shown above, sketch the solution in the (t, x) and (t, y) planes.





3. Consider the system of differential equations: $\mathbf{Y}' = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \mathbf{Y}$

(a) Find the eigenvalues to this system.
SOLUTION: From $\lambda^2 + 4 = 0$, we get $\lambda = \pm 2i$

(b) Find the general solution of this system.
SOLUTION: We need an eigenvector first, and we'll use $\lambda = 2i$:

$$-2iv_1 + 4v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 4 \\ 2i \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ i \end{bmatrix} \quad \Rightarrow \quad e^{\lambda t}\mathbf{v} = (\cos(2t) + i\sin(2t)) \begin{bmatrix} 2 \\ i \end{bmatrix}$$

Multiplying this out, we get:

$$e^{\lambda t}\mathbf{v} = \begin{bmatrix} 2\cos(2t) + 2i\sin(2t) \\ -\sin(2t) + i\cos(2t) \end{bmatrix}$$

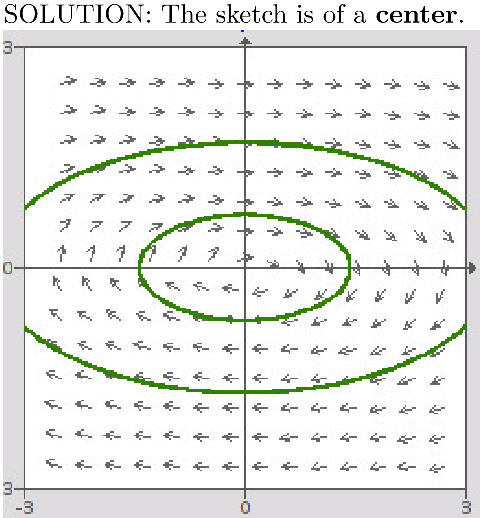
The general solution is then:

$$\mathbf{Y}(t) = C_1 \begin{bmatrix} 2\cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix}$$

(c) Find the solution that satisfies $\mathbf{Y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
SOLUTION: In this case, the equations are easy enough to solve directly. On the exam, you might need to use Cramer's Rule (it's good to remember it!)

$$\begin{array}{llll} 2C_1 & +0C_2 & = 0 \\ 0C_1 & +C_2 & = 1 \end{array} \quad \Rightarrow \quad \mathbf{Y}(t) = \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix}$$

(d) Sketch the phase plane for this system.
SOLUTION: The sketch is of a **center**.



4. Consider the family of systems of differential equations given by:

$$(a) \quad \mathbf{Y}' = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix} \mathbf{Y} \qquad\qquad (b) \quad \mathbf{Y}' = \begin{bmatrix} a & 1 \\ a & a \end{bmatrix} \mathbf{Y}$$

Use the Poincare classfication to determine how the equilibrium changes. Begin by drawing the trace-determinant plane, then track where you are on the graph.

(a) For (a), $\text{Tr}(A) = 2a$, $\det(A) = a^2 + 1$ and $\Delta = (2a)^2 - 4(a^2 + 1) = -4$.
Therefore, the determinant is always positive and the discriminant is always negative- We are inside the parabola, and the classification depends only on the trace:

- If $a > 0$, the origin is a SPIRAL SOURCE.
- If $a = 0$, the origin is a CENTER.
- If $a < 0$, the origin is a SPIRAL SINK.

(b) For (b), $\text{Tr}(A) = 2a$, $\det(A) = a^2 - a = a(a-1)$ and $\Delta = (2a)^2 - 4(a^2 - a) = 4a$
Now, the position depends on the signs of each of these quantities. One way to track them is to use a number line.

$2a$	$-$	$+$	$+$
$a(a-1)$	$+$	$-$	$+$
$4a$	$-$	$+$	$+$
	$a < 0$	$0 < a < 1$	$a > 1$

We see that if $a < 0$, then we are in the upper left quadrant, inside the parabola (SPIRAL SINK). we note that if $a = 0$, then everything is zero, and we have "UNIFORM MOTION". If $0 < a < 1$, the determinant is negative, and the origin is a SADDLE. If $a = 1$, we have a line of unstable fixed points, and finally if $a > 1$, we have a SOURCE.

5. Given the mass-spring system with unit mass ($m = 1$), spring constant $k > 0$ and damping $b \geq 0$, write the second order linear homogeneous differential equation describing the motion of the mass.
SOLUTION: $y'' + by' + ky = 0$

(a) Convert the differential equation into a system of first order equations in the form $\mathbf{Y}' = A\mathbf{Y}$ (that is, find A).

$$\mathbf{Y}' = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \mathbf{Y}$$

(b) Continuing with the previous question, find the eigenvalues of A.
SOLUTION: The characteristic equation is $\lambda^2 + b\lambda + k = 0$, so the eigenvalues are given by the quadratic formula:

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2 - 4k}}{2}$$

(c) Continuing, state all the values of b, k so that the system has complex eigenvalues.
SOLUTION: To have complex eigenvalues, the discriminant must be negative: $b^2 - 4k < 0$, or $b^2 < 4k$.

(d) State all the values of b, k for which the system has a saddle at the origin.
SOLUTION: To have a saddle at the origin, the determinant must be negative, or $k < 0$. However, this is a spring constant, so k will never be negative. Therefore, the origin is never a saddle.

6. Below is a coupled pair of second order equations in x, y. Convert it to a linear system of four first order equations. You might let u_1, u_2, u_3, u_4 be your variables.

$$\begin{aligned} x'' + 2x' - y' + x + 4y &= 0 \\ y'' + y' + 8y - 3x &= 0 \end{aligned}$$

SOLUTION: Let $u_1 = x$, $u_2 = x'$, $u_3 = y$ and $u_4 = y'$. We then write the system of DEs for u_i:

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= x'' = -x - 4y - 2x' + y' = -u_1 - 2u_2 - 4u_3 + u_4 \\ u_3' &= u_4 \\ u_4' &= y'' = 3x - 8y - y' = 3u_1 - 8u_3 - y_4 \end{aligned}$$

It's not necessary, but we could write this system as:

$$\mathbf{u}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & -4 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -8 & -1 \end{bmatrix} \mathbf{u}$$

7. Solve: $\mathbf{Y}' = A\mathbf{Y}, \mathbf{Y}(0)$ shown.

   (a) $A = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}$, $\mathbf{Y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Sketch the phase plane for this one.

   SOLUTION: $Tr(A) = -7$ and $\det(A) = 10$, so $\lambda^2 + 7\lambda + 10 = 0$ gives us $(\lambda + 2)(\lambda + 5) = 0$, or $\lambda = -2, -5$ (the origin is a SINK). Now compute eigenvectors:

   - For $\lambda = -2$, we have $(-4 - (-2))v_1 + v_2 = 0$, or $-2v_1 + v_2 = 0$.
     One eigenvector would be $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
   - For $\lambda = -5$, we have $(-4 - (-5))v_1 + v_2 = 0$, or $v_1 + v_2 = 0$.
     One eigenvector would be $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   The general solution is now

$$\mathbf{Y}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

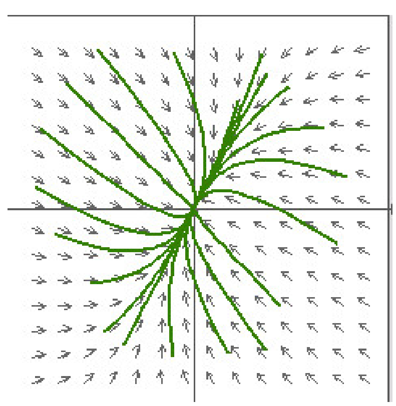
   To get the initial condition,

$$\begin{aligned} C_1 - C_2 &= 1 \\ 2C_1 + C_2 &= 1 \end{aligned} \quad \Rightarrow \quad C_1 = \frac{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}} = \frac{2}{3}, \quad C_2 = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}} = \frac{-1}{3}$$

   The general solution is now

$$\mathbf{Y}(t) = \frac{2}{3} e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{3} e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   For the phase plane, I would be looking for your eigenvectors and the solution curves should bend towards the first eigenvector (the one with the "not-so-negative" exponent).



(b) $A = \begin{bmatrix} -4 & 4 \\ -1 & 0 \end{bmatrix}$, $\mathbf{Y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

We follow the same procedure. In this case, $\mathrm{Tr}(A) = -4$ and $\det(A) = 4$, so that

$$\lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \quad (\lambda+2)^2 = 0 \quad \Rightarrow \quad \lambda = -2, -2$$

We have a double eigenvalue. If we solved for the eigenvectors, we would just get one: $(-4+2)v_1 + 4v_2 = 0$, or $-v_1 + 2v_2 = 0$. But its not needed this time. We just need to solve for $\mathbf{v}_1$, which we said would be:

$$\begin{array}{ll} (-4+2)(1) + 4(2) & = v_{11} \\ -1(1) + (0+2)(2) & = v_{12} \end{array} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

The solution is:

$$\mathbf{Y}(t) = \mathrm{e}^{-2t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right)$$

It wasn't asked, but it is good practice to sketch the direction field (in the phase plane) for this solution. For that, we need an eigenvector, and that is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. After that, look at directions for the flow using $(\pm 1, 0)$ and $(0, \pm 1)$ for some handy points. Here's generally what it should look like: