Exam 1 Review

The material will be Sections 1.1-1.9 (exc. 1.7). If you'd like to have more review questions, the textbook has a good set at the end of Chapter 1.

This part of the course has looked at techniques for constructing and analyzing first order differential equations, and building models based on first order DEs.

1. What is a differential equation (as a definition, and in terms of explaining the definition)? What does it represent, and what does it mean to "solve" it? Understand that there are three ways of understanding a differential equation (Graphical, Numerical, and Analytic).

2. Other vocabulary terms:

   Ordinary DE, Partial DE, *order* of a DE, linear first order DE, nonlinear DE, IVP, particular (versus general) solution, Autonomous DE, separable DE.

3. Models:

   Exponential growth, The logistic population model (or logistic growth), Newton's Law of Cooling (state in words and as a model), Free fall (use the book's quadratic assumption on air resistance), Tank mixing (set it up), Savings model (think of it like population or tank mixing, p 29).

4. Theory: The Existence and Uniqueness theorem. State what the theorem says, be able to apply it to a DE, and give the results. Understand the implications of the E & U Theorem- In particular, if a DE satisfies the theorem everywhere in the plane, then solutions cannot intersect (in the (t, y) plane).

   Often, we can build an infinite number of solutions (for those DEs that allow it) by hooking up equilibrium solutions with other solutions. We saw that with # 11, and # 10 does the same kind of thing in 1.5. A question like this will be found in the sample.

5. Qualitative Analysis (or Graphical Analysis):

   There are different levels of analysis possible, depending on the form of the DE:

   - $y' = f(t, y)$. From a slope field, we can sketch out possible solutions, and from that, give some estimates on the overall behavior.
   - $y' = f(t)$. The slopes here depend only on t, not on y- On any vertical line in the direction field, all the slopes are the same. Different solutions are vertical shifts of one another.
   - $y' = f(y)$. Here, we can build and analyze the slopes using a new graph: (y, y'). On this graph, we can create the **phase line** and transfer that information to the direction field (the graph of (t, y)). We can also determine the concavity of $y(t)$ from the (y, y') graph. Be able to locate and classify all equilibria (sinks, sources,

nodes) graphically and using $\frac{df}{dy}$. New solutions can be constructed from other solutions by shifting them right or left (for example, see p 41, Figure 1.22).

You might think about how you might create a differential equation based on a sketch of what you want in the (y, y') plane.

6. Numerical Analysis

   Be able to state Euler's Method. In particular, understand where it comes from, and be able to discuss how to derive the formula.

   Be able to compute a step or two using Euler's Method.

7. The Analytic Technique: We have two types of DEs that we can solve: separable and linear.

   (a) Separable: $\dfrac{dy}{dt} = f(y)g(t)$. Then

   $$\frac{1}{f(y)}\,dy = g(t)\,dt \quad \Rightarrow \quad \int \frac{1}{f(y)}\,dy = \int g(t)\,dt$$

   This will typically give you $y(t)$ *implicitly*. When asked, be able to solve for $y(t)$ to get and *explicit* solution.

   There are two special cases: $y' = f(y)$ and $y' = g(t)$.

   (b) Linear: Be sure to have the form $y' + a(t)y = f(t)$. Then compute the **integrating factor**:

   $$\mu(t) = \mathrm{e}^{\int a(t)\,dt}$$

   The reason we compute this is because (and you should double check this):

   $$(\mu(t)y(t))' = \mu(t)y' + a(t)\mu(t)y$$

   Therefore, we multiply both sides of the linear DE by $\mu(t)$ and we get:

   $$(\mu(t)y(t))' = \mu(t)f(t)$$

   Now, integrate both sides with respect to t and divide my $\mu(t)$ to get $y(t)$.

   (c) Second way of solving a linear equation: $y' = -a(t)y + g(t)$.

   We break up the solution by first solving the equation when $g(t) = 0$:

   $$y' = -a(t)y$$

   which is separable. We call this solution the *homogeneous part of the solution*, $y_h(t)$. We then guess at the full solution by taking functions of the same "form" as g. We find this part of the solution by substitution (this is $y_p(t)$). The full general solution is then given as

   $$y_h(t) + y_p(t)$$

8. Solving by Substitution

   In problems like Appendix A, the DE is not in the right form to be solved (separable or linear), but with the suggested substition, becomes either separable or linear. I will always give the suggested substition for these.

Example Questions

1. True or False, and explain:

   (a) If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.

   (b) Let $\frac{dy}{dt} = 1+y^2$. The Existence and Uniqueness theorem tells us that the solution (for any initial value) will be valid for all t. If true, say why. If False, solve the DE and find the interval on which the solution exists.

   (c) Let $y' = f(y)$. It is possible to have two stable equilibrium with no other equilibrium between them (you may assume df/dy is continuous- Consider what this means in the (y, y') plane).

   (d) If $y' = \cos(y)$, then the solutions are periodic.

   (e) All autonomous equations are separable.

2. Give either the general solution, or if an initial value is given, compute the particular solution.

   (a) $\frac{dy}{dx} = \frac{x^2 - 2y}{x}$

   (b) $\frac{dy}{dx} = 2\cos(3x) \qquad y(0) = 2$

   (c) $y' - \frac{1}{2}y = 0 \qquad y(0) = 200$. Additionally, give the interval on which the solution is valid.

   (d) $y' = (1 - 2x)y^2 \qquad y(0) = -1/6$. Additionally, give the interval on which the solution is valid.

   (e) $y' - \frac{1}{2}y = e^{2t} \qquad y(0) = 1$

   (f) $y' = \frac{1}{2}y(3 - y)$

   (g) $\sin(2t)\ dt + \cos(3y)\ dy = 0$

   (h) $y' = ty^2$

   (i) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$

   (j) $x^3\frac{dy}{dx} = 1 - 2x^2y$.

   (k) $y' = 2(1 + x)(1 + y^2), y(0) = 0$

3. Find a linear differential equation of the form: $y' + a(t)y = g(t)$ if we want the following to be a solution: $y(t) = t + \frac{C}{t^2}$. Hint: $a(t)$ should be found so that $y' + a(t)y$ has no C. Then $g(t)$ would be given by $y' + a(t)y$. After you find it, try solving it to be sure!

4. Graphically, what does the Existence and Uniqueness theorem mean? For example, if you were looking at a set of solutions, what would indicate that the E & U theorem did not apply?

5. Give a short derivation of Euler's Method. That is, given $y' = f(t, y)$ is the DE, and we start at $y(t_0) = y_0$ with step size Δt, show how we get the next value, y_1. In particular, what is the assumption behind Euler's Method?

6. Use Euler's Method to compute y_1, y_2 if $y(0) = 1$, $\Delta t = 1$ and $y' = (y + 1)(t + 1)$. Do you think y_1, y_2 will be very close to the actual solution? Why or why not?

7. Construct a model of population using the logistic equation, if the initial growth rate is approximately 2, and the carrying capacity of the environment is approximately 100.

8. Suppose I have a cup of coffee. When I enter a room (70°F, constant), my coffee is at 110° and one hour later, it is at 80°. Find a function that gives the temperature of my coffee at all $t \geq 0$.

9. Let $y' = y^{1/3}$, $y(0) = 0$. Find two solutions to the IVP. Does this violate the Existence and Uniqueness Theorem?

10. A salty brine is being pumped into a tank. Suppose the salt is being pumped in at 1/2 pound per gallon, and the brine is coming in at 2 gallons per minute. The brine is well mixed and is being pumped out at 2 gallons per minute. Initially, the tank has 100 gallons of pure water.

    (a) Find an IVP that will model the amount of salt in the tank at time t.

    (b) Without actually solving, if $S(t)$ is the amount of salt in the tank at time t, what should $S(t)$ be as $t \to \infty$?

    (c) Solve the IVP.

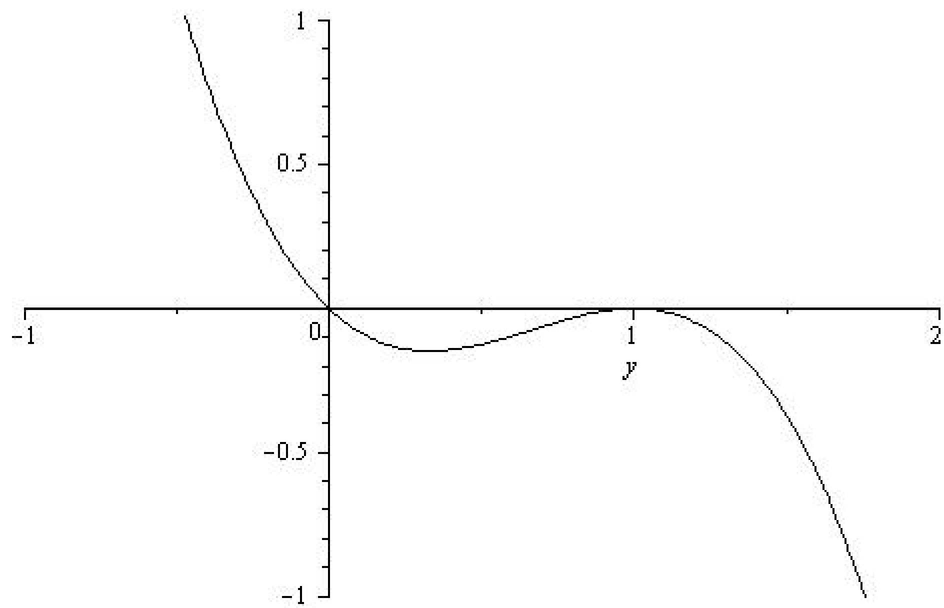
11. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v^2$, find the initial value problem for the velocity at time t. In the (t, y) plane, draw several solution curves. How would you go about solving this analytically (you don't need to do that, but tell me how).

12. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of k dollars per month, set up a model for how much you owe at time t in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

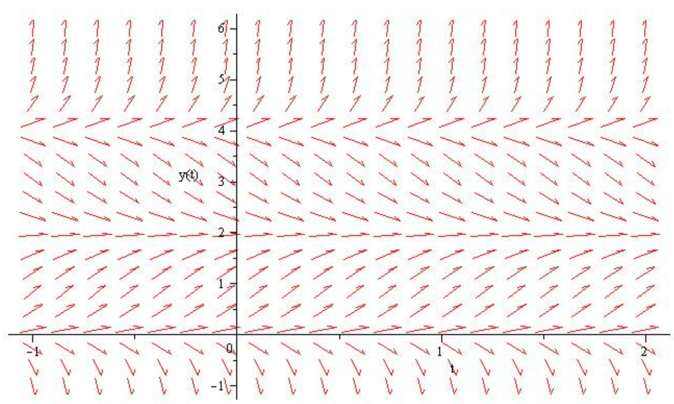
13. Show that the IVP $xy' = y - 1$, $y(0) = 2$ has no solution. (Note: Part of the question is to think about how to show that the IVP has no solution).

14. Suppose that a certain population grows at a rate proportional to the square root of the population. Assume that the population is initially 400 (which is 20^2), and that one year later, the population is 625 (which is 25^2). Determine the time in which the population reaches 10000 (which is 100^2).

15. Consider the sketch below of $F(y)$, and the differential equation $y' = F(y)$.
    (a) Find and classify the equilibrium.
    (b) Find intervals (in y) on which $y(t)$ is concave up.
    (c) Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down.
    (d) Find an appropriate polynomial for $F(y)$.



16. Given the direction field to the right, write a differential equation that is consisten with it.



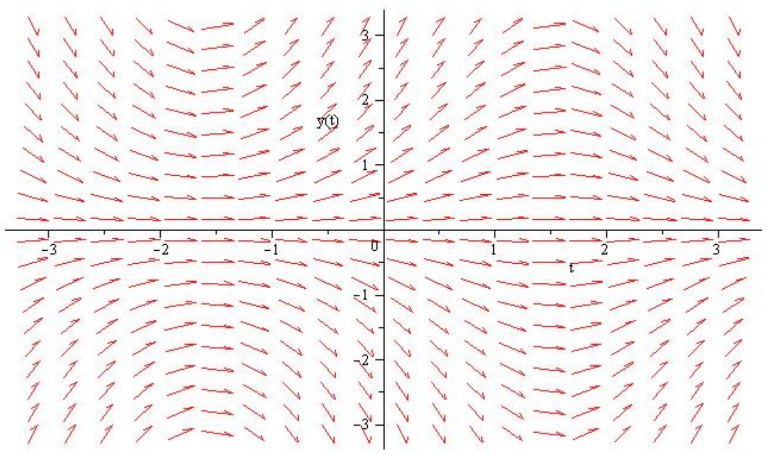
17. Consider the direction field below, and answer the following questions:

   (a) Is the DE possibly of the form $y' = f(t)$?

   (b) Is the DE possible of the form $y' = f(y)$?

   (c) Is there an equilibrium solution? (If so, state it):

   (d) Draw the solution corresponding to $y(-1) = 1$.



18. For the following differential equations, show that, with the suggested substitution, that the DE becomes either linear or separable. You do NOT need to go further than that.

   (a) $(x + y)\,dx - (x - y)\,dy = 0$ Substitute $w = y/x$

   (b) $\dfrac{dy}{dx} - \dfrac{3}{2x}y = \dfrac{2x}{y}$ Substitute $w = y^2$.

19. Evaluate the following integrals:

$$\text{(a)} \int \frac{x}{(x-1)(2-x)}\,dx \qquad \text{(b) } \mathrm{e}^{-3\int \frac{1}{t}\,dt}$$

20. For the following, write out what the partial fraction expansion would be (but do NOT solve for the coefficients!)

$$\text{(a)} \int \frac{x^2-1}{(x+1)(x^2+1)}\,dx \qquad \text{(b)} \int \frac{3x}{(x+1)^2(x+2)}\,dx$$