Solutions to the Review Questions

Short Answer/True or False

1. True or False, and explain:

   (a) If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.
   False: This is an isocline associated with a slope of zero, and furthermore, $y = -2t$ is not a solution, and it is not a constant.

   (b) Let $\frac{dy}{dt} = 1 + y^2$. The Existence and Uniqueness theorem tells us that the solution (for any initial value) will be valid for all t. (If true, say why. If False, solve the DE).
   False. The E & U Theorem tells that a unique solution will exist for any initial condition (since $1 + y^2$ and $2y$ are continuous everywhere), but it does not say on what interval the solution will exist. For example, if we take $y(0) = y_0$ and solve, we get:

$$\int \frac{dy}{1+y^2} = \int dt \quad \Rightarrow \quad \tan^{-1}(y) = t + C \quad \Rightarrow \quad C = \tan^{-1}(y_0)$$

   Therefore,

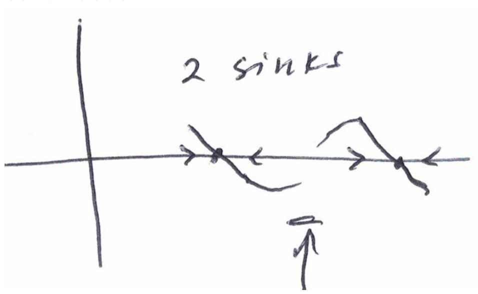
$$y = \tan(t + \tan^{-1}(y_0))$$

   where

$$-\frac{\pi}{2} < t + \tan^{-1}(y_0) < \frac{\pi}{2}$$

   (so that the tangent function is invertible).

   (c) Let $y' = f(y)$. It is possible to have two stable equilibrium with no other equilibrium between them (you may assume df/dy is continuous- Consider what this means in the (y, y') plane).
   SOLUTION: (NOTE: I should have used the term SINK rather than STABLE- Hope that didn't confuse you! And we didn't need f' to be continuous, but rather f (copy and paste erros)). I meant for you to some graphical analysis to solve this- See the graph below. The answer is no, not if f is continuous.



   (d) If $y' = \cos(y)$, then the solutions are periodic.
   FALSE. A function y is periodic if it is periodic in t, and once a function increases (for example), it cannot decrease again (since the slopes along any horizontal line are constant).

   (e) All autonomous equations are separable.
   True. We can write $y' = f(y) \cdot 1$, which is separable and $\displaystyle\int \frac{dy}{f(y)} = \int dt$.

2. Give the general solution, or solve the IVP:

   (a) $\displaystyle\frac{dy}{dx} = \frac{x^2 - 2y}{x}$
   Linear: $y' + \frac{2}{x}y = x$ Solve with an integrating factor of x^2 to get:

$$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

(b) $\dfrac{dy}{dx} = 2\cos(3x)$, $y(0) = 2$.
SOLUTION: Separable- Integrate directly then solve for C.

$$y(x) = \frac{2}{3}\sin(3x) + 2$$

(c) $y' = \frac{1}{2}y$, $y(0) = 2$.
SOLUTION: Separable- $y(t) = 200\mathrm{e}^{t/2}$. The solution is valid for all time.

(d) $y' = (1-2x)y^2$ with $y(0) = -1/6$.
SOLUTION: Separable- Separate variables and integrate on both sides:

$$-\frac{1}{y} = x - x^2 + C \qquad \Rightarrow \qquad 6 = 0 + 0 + C \qquad \Rightarrow \qquad y = \frac{1}{x^2 - x - 6} = \frac{1}{(x+2)(x-3)}$$

The function y has vertical asymptotes at $t = -2$ and $t = 3$ which cuts time into three intervals. We choose $-2 < t < 3$ since we need the interval on which $t = 0$ is included.

(e) $y' - \frac{1}{2}y = \mathrm{e}^{2t} \qquad y(0) = 1$
This is linear (but not separable). $y(t) = \frac{2}{3}\mathrm{e}^{2t} + \frac{1}{3}\mathrm{e}^{(1/2)t}$

(f) $y' = \frac{1}{2}y(3-y)$
Autonomous (and separable). Integrate using partial fractions:

$$\int \frac{1}{y(3-y)}\,dy = \frac{1}{2}\int dt \qquad \Rightarrow \qquad \frac{1}{3}\ln|y| - \frac{1}{3}\ln|3-y| = \frac{1}{2}t + C \qquad \Rightarrow \qquad \ln\left|\frac{y}{3-y}\right| = \frac{3}{2}t + C_2$$

Therefore,

$$\frac{y}{3-y} = A\mathrm{e}^{3t/2} \qquad \Rightarrow \qquad y(t) = \frac{3A\mathrm{e}^{3t/2}}{1 + A\mathrm{e}^{3t/2}}$$

Notice that you could divide top and bottom by the expontial term to end up with:

$$y(t) = \frac{3}{1 + B\mathrm{e}^{-(3/2)t}}$$

which some people prefer (its OK to leave your answer in the other form).

(g) $\sin(2t)\,dt + \cos(3y)\,dy = 0$
Separable:

$$\cos(3y)\,dy = -\sin(2t)\,dt \qquad \Rightarrow \qquad \int \cos(3y)\,dy = -\int \sin(2t)\,dt$$

so that: $\frac{1}{3}\sin(3y) = \frac{1}{2}\cos(2t) + C$. Leave your answer in that form.

(h) $y' = ty^2$
Separable: $y = \dfrac{1}{-(1/2)t^2 + C}$

(i) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$
SOLUTION: Try to get it into factored form, then we can separate variables and solve.

$$y' = \frac{-2y(xy+1)}{2x(xy+1)} = \frac{-y}{x} \qquad \Rightarrow \qquad \int \frac{dy}{y} = -\int \frac{dx}{x} \qquad \Rightarrow \qquad \ln|y| = -\ln|x| + C \qquad \Rightarrow \qquad y = \frac{A}{x}$$

(j) $x^3\dfrac{dy}{dx} = 1 - 2x^2y$.
SOLUTION: This one doesn't factor, but it is linear. We can see that by expressing it in standard form:

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^3}$$

The integrating factor is x^2:

$$(x^2y)' = \int \frac{1}{x}\,dx \qquad \Rightarrow \qquad x^2y = \ln|x| + C \qquad \Rightarrow \qquad y = \frac{\ln|x| + C}{x^2}$$

(k) $y' = 2(1+x)(1+y^2), y(0) = 0$

SOLUTION: This is also separable:

$$\int \frac{dy}{1+y^2} = \int 2(1+x)\,dx \quad \Rightarrow \quad \arctan(y) = (1+x)^2 + C$$

Solving now for C, we see that $0 = 1 + C$, or $C = -1$. Continuing,

$$y = \tan((1+x)^2 - 1)$$

3. Construct a linear first order differential equation whose general solution is given by: $y = t + C/t^2$.

   SOLUTION: Examing y and y', we see that:

   $$y' + \frac{2}{t}y = 3$$

   Solving this from scratch, we get the desired solution.

4. Graphically, what does the Existence and Uniqueness theorem mean? For example, if you were looking at a set of solutions, what would indicate that the E & U theorem did not apply?

   Existence and Uniqueness will mean that two solution curves must never intersect.

5. Give a short derivation of Euler's Method. That is, given $y' = f(t, y)$ is the DE, and we start at $y(t_0) = y_0$ with step size Δt, show how we get the next value, y_1. In particular, what is the assumption behind Euler's Method?

   SOLUTION: (You should draw a sketch like we did in class). The main assumption is that $f(t_0, y_0)$ will stay constant on the interval $[t_0, t_0 + \Delta t]$ (which it almost never will). In that case,

   $$y_1 = y_0 + f(t_0, y_0)\Delta t$$

   or in general,

   $$y_{k+1} = y_k + f(t_k, y_k)\Delta t$$

6. Use Euler's Method to compute y_1, y_2 if $y(0) = 1$, $\Delta t = 1$ and $y' = (y+1)(t+1)$. Do you think y_1, y_2 will be very close to the actual solution? Why or why not?

   SOLUTION: To begin with, the actual solution will not be close to y_1, y_2 because Δt is huge (value is 1). However, this is typically what we do when computing by hand:

   $$y_1 = 1 + (1+1)(0+1) \cdot 1 = 3$$

   $$y_2 = 3 + (3+1)(1+1) \cdot 1 = 11$$

7. Construct a model of population using the logistic equation, if the initial growth rate is approximately 2, and the carrying capacity of the environment is approximately 100.

   SOLUTION: The general model is given by:

   $$y' = ky\left(1 - \frac{y}{N}\right) = 2y\left(1 - \frac{y}{100}\right)$$

8. Suppose I have a cup of coffee. When I enter a room (70°F, constant), my coffee is at 110° and one hour later, it is at 80°. Find a function that gives the temperature of my coffee at all $t \geq 0$.

   SOLUTION: Newton's Law of Cooling states that: The rate of change of the temp of a body is proportional to the difference between the temp of the body and the environment. Thus, we can write:

   $$y' = -k(y - T)$$

   where $k > 0$ and T is constant (environmental temp). We can solve this generally for y:

   $$\int \frac{dy}{y - T} = -k \int dt \quad \Rightarrow \quad \ln|y - T| = -kt + C \quad \Rightarrow \quad y = Ae^{-kt} + T$$

Substituting in our room temp, then solve for the initial coffee temp:

$$y = Ae^{-kt} + 70 \quad \Rightarrow \quad 110 = A + 70 \quad \Rightarrow \quad A = 40$$

Now we have:

$$y = 40e^{-kt} + 70$$

Finally, we need the other point in order to solve for k: Use $t = 1$ for one hour, and:

$$80 = 40e^{-k} + 70 \quad \Rightarrow \quad \frac{1}{4} = e^{-k} \quad \Rightarrow \quad \ln(4^{-1}) = -k \quad \Rightarrow \quad \ln(4) = k$$

so that you could write:

$$y(t) = 40e^{-\ln(4)t} + 70$$

NOTE: Its fine if you write $e^{\ln(1/4)\,t}$ in place of $e^{-\ln(4)\,t}$

9. Let $y' = y^{1/3}$, $y(0) = 0$. Find two solutions to the IVP. Does this violate the Existence and Uniqueness Theorem?

   SOLUTION: Using our usual method,

$$\int y^{-1/3}\,dy = \int\,dt \quad \Rightarrow \quad \frac{3}{2}y^{2/3} = t + C \quad \Rightarrow \quad y^{2/3} = \frac{2}{3}t + C_2 \quad \Rightarrow \quad y(t) = \left(\frac{2}{3}t\right)^{3/2}$$

   Notice that $y(t) = 0$ is also a solution to the IVP (this is the equilibrium solution, and is often overlooked!).

10. A salty brine is being pumped into a tank. Suppose the salt is being pumped in at 1/2 pound per gallon, and the brine is coming in at 2 gallons per minute. The brine is well mixed and is being pumped out at 2 gallons per minute. Initially, the tank has 100 gallons of pure water.

    (a) Find an IVP that will model the amount of salt in the tank at time t.
    SOLUTION: Writing it like a linear DE:

$$\frac{dS}{dt} = -\frac{1}{50}S + 1$$

    (b) Without actually solving, if $S(t)$ is the amount of salt in the tank at time t, what should $S(t)$ be as $t \to \infty$?
    SOLUTION: Over time, the concentration should become the incoming concentration of 1/2 lb per gallon- With 100 gallons, that means $S(t) \to 50$.

    (c) Solve the IVP.
    SOLUTION:

$$S(t) = 50 - 50e^{-t/50}$$

11. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v^2$, find the initial value problem for the velocity at time t. In the (t, y) plane, draw several solution curves. How would you go about solving this analytically (you don't need to do that, but tell me how).

    SOLUTION: TYPO: I meant (t, v) plane rather than (t, y) plane.

    Recall that the general model was: $mv' = mg - kv$, or in this case, $mv' = mg - kv^2$. Subsituting values,

$$v' = 9.8 - \frac{1}{2}v^2, \quad v(0) = 0$$

    This is our usual type of autonomous DE. In the (v, v') plane, we have an upside down parabola, with a source at the negative equilibrium solution and a sink at the positive equilibrium. Since it is autonomous, we could solve this using separation of variables (followed by partial fractions).

12. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of k dollars per month, set up a model for how much you owe at time t in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

    Define $S(t)$ to be the amount owed after t years (note that k is given in months, so $12k$ is needed for years). Then:

    $$\frac{dS}{dt} = \frac{1}{20}S - 12k \qquad S(0) = 10 \qquad \text{(S is in thousands)}$$

    Solving the IVP, we get:

    $$S(t) = \mathrm{e}^{t/20}(10 - 240k) + 240k$$

    Finding k so that $S(10) = 0$, we get:

    $$0 = \mathrm{e}^{1/2}(10 - 240k) + 240k \quad \Rightarrow \quad k \approx 0.1058$$

    This is in thousands, so our monthly payment is \$105.80, or annual payment is \$1270. Notice that this means we pay back \$15420 for the loan.

13. Show that the IVP $xy' = y - 1$, $y(0) = 2$ has no solution. (Note: Part of the question is to think about how to show that the IVP has no solution).

    SOLUTION: If we try to solve it using separation of variables, we get:

    $$\int \frac{dy}{y-1} = \int \frac{dx}{x} \quad \Rightarrow \quad y(x) = Ax + 1$$

    Notice that $y(0) = 1$ for all values of A, therefore we cannot determine a value of A that will solve the IVP- The IVP has no solution.

    We might note that

    $$\frac{dy}{dx} = f(x, y) = \frac{y-1}{x}$$

    so that, by the Existence and Uniqueness theorem, we have to "watch out" at $x = 0$.

14. Suppose that a certain population grows at a rate proportional to the square root of the population. Assume that the population is initially 400 (which is 20^2), and that one year later, the population is 625 (which is 25^2). Determine the time in which the population reaches 10000 (which is 100^2).

    SOLUTION: If the rate of change is proportional to the square root of the population, then:

    $$\frac{dP}{dt} = k\sqrt{P} = kP^{1/2} \quad \Rightarrow \quad P(t) = (2kt + C)^2$$

    Putting in the intial condition,

    $$20^2 = (0 + C_2)^2 \quad \Rightarrow \quad C_2 = 20$$

    And the condition at time 1:

    $$25^2 = (2k + 20)^2 \quad \Rightarrow \quad P(t) = (5t + 20)^2$$

    Finally, solve for t that makes $P = 100^2$ (find that $t = 16$).

15. Consider the sketch below of $F(y)$, and the differential equation $y' = F(y)$.

    (a) Find and classify the equilibrium.

    SOLUTION: From the sketch given, $y = 0$ is asymptotically stable and $y = 1$ is semistable.

    (b) Find intervals (in y) on which $y(t)$ is concave up.

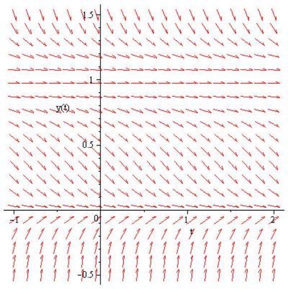
    SOLUTION: Examine the intervals $y < 0$, $0 < y < 1/3$, $1/3 < y < 1$ and $y > 1$ separately. The function y will be concave up when dF/dy and F both have the same sign- This happens when F is either increasing and positive (which happens nowhere) or decreasing and negative:

    $$0 < y < \frac{1}{3} \qquad y > 1$$

(c) Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. See the figure below.

(d) Find an appropriate polynomial for $F(y)$.
SOLUTION: One example is

$$y' = -y(y-1)^2$$



16. From the direction field, we see three equilibria at $y = 0, y = 2, y = 4$. Sketching the phase line, we see that one possibility is:

$$y' = y(y-2)(y-4)$$

17. Consider the direction field below, and answer the following questions:

(a) Is the DE possibly of the form $y' = f(t)$?
SOLUTION: No. The isoclines would be vertical (consider, for example, a vertical line at $t = -3$; the slopes are clearly not equal).

(b) Is the DE possible of the form $y' = f(y)$?
SOLUTION: No. The isoclines would be horizontal (for example, look at a horizontal line at $y = 1$- Some slopes are zero, others are not).

(c) Is there an equilibrium solution? (If so, state it):
SOLUTION: Yes- At $y = 0$.

(d) Draw the solution corresponding to $y(-1) = 1$.
SOLUTION: Just draw a curve consistent with the arrows shown.

18. Substitution problems:

(a) We can write the expression as:

$$\frac{dy}{dx} = \frac{1+w}{1-w}$$

To substitute for dy/dx, we write: $y = xw$ and differentiate using the product rule, or $y' = w + xw'$. Now:

$$w + x\frac{dw}{dx} = \frac{1+w}{1-w} \qquad \Rightarrow \qquad xw' = \frac{1+w^2}{1-w}$$

and this is clearly separable.

(b) If we multiply by y first,

$$yy' - \frac{3}{2x}y^2 = 2x$$

Substitute $w = y^2$, and for the derivative, $w' = 2yy'$, or $yy' = \frac{1}{2}w'$:

$$\frac{1}{2}w' - \frac{3}{2x}w = 2x \qquad \Rightarrow \qquad w' - \frac{3}{x}w = 4x$$

And this is linear.

19. Integral practice:

(a)

$$\int \frac{x}{(x-1)(2-x)}\,dx = \int \frac{1}{x-1} + \frac{2}{2-x}\,dx = \ln|x-1| - 2\ln|2-x| + C$$

(b)

$$e^{-3\ln|t|} = \frac{1}{t^3}$$

20. For partial fractions (we don't need to include the integral signs):

(a) $\dfrac{x^2-1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

(b) $\dfrac{3x}{(x+1)^2(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2}$