Worked Examples from Class

There was still some question from #3 in 6.2. We are given $g_a(t)$ in piecewise form and we want to determine its Laplace transform. For specificity, we will find the transform using the table. Here is the function:

$$g_a(t) = \begin{cases} t/a & \text{if } t \le a \\ 1 & \text{if } t > a \end{cases}$$

To put this in a form from which we can use the table, we need to write $g_a(t)$ using the Heaviside function. From the "on-off" switch, we write:

$$g_a(t) = (u_0(t) - u_a(t))\frac{t}{a} + u_a(t)$$

Since we work with $t \ge 0$, we can write $u_0(t) = 1$, and re-writing g_a, we get:

$$g_a(t) = \frac{t}{a} + u_a(t)\left(1 - \frac{t}{a}\right)$$

Taking the Laplace transform using the table, the first entry uses table entry 2, and the second one uses #10. More specifically, to use #10, we write:

$$u_a(t)f(t-a) = u_a(t)\left(1 - \frac{t}{a}\right)$$

From this, we need to determine $f(t)$ so that we can compute $F(s)$. We see that:

$$f(t-a) = \left(1 - \frac{t}{a}\right) \quad \Rightarrow \quad f(t) = \left(1 - \frac{t+a}{a}\right) = 1 - \frac{t}{a} + 1 = -\frac{t}{a}$$

Therefore, $F(s) = -\frac{1}{a}\frac{1}{s^2}$, or all together,

$$G_a(s) = \frac{1}{as^2} - \frac{e^{-as}}{as^2}$$

Extra Example

Suppose that f is given below in piecewise form. Write it in terms of the Heaviside function, then find its Laplace transform.

$$f(t) = \begin{cases} 3 & \text{if } t < 1 \\ t^2 & \text{if } 1 \le t \le 4 \\ 3t+1 & \text{if } 4 < t \le 5 \\ e^{-2t} & \text{if } t > 5 \end{cases}$$

SOLUTION: We write this first using the "on-off" switches, one for each function:

$$f(t) = (1 - u_1(t))3 + (u_1(t) - u_4(t))t^2 + (u_4(t) - u_5(t))(3t+1) + u_5(t)e^{-2t}$$

To take the Laplace transform, we'll re-write in the form $u_c(t)f(t-c)$:

$$f(t) = 3 + u_1(t)(t^2-3) + u_4(t)(1+3t-t^2) + u_5(t)(\mathrm{e}^{-2t} - 3t - 1)$$

Now, for each in turn, we write these in the form $u_c(t)f(t-c)$, then find $f(t)$ (then find $F(s)$).

$$f(t-1) = t^2 - 3 \quad \Rightarrow \quad f(t) = (t+1)^2 - 3 = t^2 + 2t - 2 \quad \Rightarrow \quad F(s) = \frac{2}{s^3} + \frac{2}{s^2} - \frac{2}{s}$$

For the next function,

$$f(t-4) = 1 + 3t - t^2 \quad \Rightarrow \quad f(t) = -t^2 - 5t - 3 \quad \Rightarrow \quad F(s) = -\frac{2}{s^3} - \frac{5}{s^2} - \frac{3}{s}$$

And for the last Heaviside,

$$f(t-5) = \mathrm{e}^{-2t} - 3t - 1 \quad \Rightarrow \quad f(t) = \mathrm{e}^{-2(t+5)} - 3t - 16 \quad \Rightarrow \quad F(s) = \frac{\mathrm{e}^{-10}}{s+2} - \frac{3}{s^2} - \frac{16}{s}$$

Now we put it all together-

$$\frac{3}{s} + \mathrm{e}^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} - \frac{2}{s}\right) + \mathrm{e}^{-4s}\left(-\frac{2}{s^3} - \frac{5}{s^2} - \frac{3}{s}\right) + \mathrm{e}^{-5s}\left(\frac{\mathrm{e}^{-10}}{s+2} - \frac{3}{s^2} - \frac{16}{s}\right)$$

OK, we got a little carried away with that one. Here is the last example we were computing in class:

Example

Solve the following using the Laplace transform:

$$y'' + 2y' + 5y = 5(1 - u_7(t)), \quad y(0) = 0, \quad y'(0) = 0$$

Taking the Laplace transform of both sides,

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + 5\mathcal{L}(y) = \mathcal{L}(5 - 5u_7(t))$$

$$(s^2 + 2s + 5)Y(s) = \frac{5}{s} - \frac{5\mathrm{e}^{-7s}}{s} = \frac{5 - 5\mathrm{e}^{-7s}}{s}$$

so that

$$Y(s) = \frac{5 - 5\mathrm{e}^{-7s}}{s(s^2+2s+5)} = (1 - \mathrm{e}^{-7s})\frac{5}{s(s^2+2s+5)}$$

Side Remark

If you are inverting something of the form:

$$(\mathrm{e}^{-c_1 s} - \mathrm{e}^{-c_2 s})H(s)$$

then the solution is:

$$u_{c_1}(t)h(t-c_1) - u_{c_2}(t)h(t-c_2)$$

Therefore, we only need to focus on inverting $H(s)$.

Back to the example

In this case, that means inverting the following- We've already done the partial fraction expansion below:

$$H(s) = \frac{5}{s(s^2+2s+5)} = \frac{1}{s} - \frac{s+2}{s^2+2s+5}$$

Invert $1/s$ as 1.

Invert the second one by first completing the square:

$$\frac{s+2}{s^2+2s+5} = \frac{s+2}{(s^2+2s+1)+4} = \frac{s+2}{(s+1)^2+2^2}$$

To use the table, we need $s+1$ in the numerator (for the cosine term), so we add/subtract what we need in the numerator:

$$\frac{s+2}{(s+1)^2+2^2} = \frac{s+1}{(s+1)^2+2^2} + \frac{1}{(s+1)^2+2^2}$$

Now we need the second fraction to have a "2" in the numerator- Multiply and divide by 2:

$$\frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

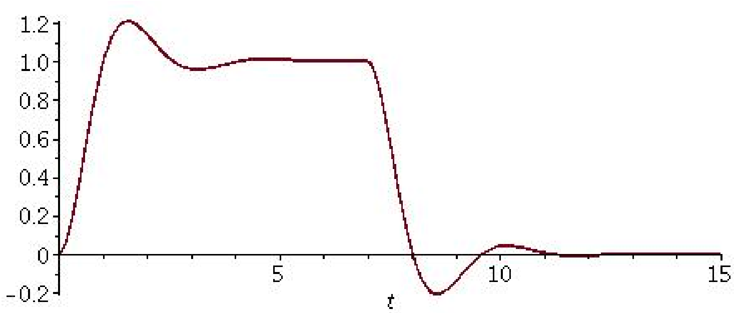
Now this is ready to invert- Using the table (Entries 5 and 6), we get:

$$h(t) = 1 - e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)$$

Now the full solution is, in terms of $h(t)$:

$$y(t) = u_0(t)h(t) - u_7(t)h(t-7) = h(t) - u_7(t)h(t-7)$$

And we can leave the solution in that form. Let's look at a graph of the solution:



Initially, we see the solution converging to the forced response of 1. At time $t = 7$, the system re-sets, and the new solution oscillates for a bit before converging to 0.

Other examples:

- Find the inverse Laplace transform:

$$\frac{s+3}{s^2+s+1}$$

SOLUTION: We'll need to complete the square in the denominator to prep the fraction to look like table entries 5 and 6:

$$\frac{s+3}{(s^2+s+1/4)+3/4} = \frac{s+3}{(s+1/2)^2+3/4}$$

To use 5/6, we need $s+1/2$ in the numerator, not $s+3$, so I add and subtract to get the form that we need:

$$\frac{s+1/2}{(s+1/2)^2+3/4} + \frac{5}{2}\frac{1}{(s+1/2)^2+3/4}$$

In the right-hand expression, we need $\sqrt{3/4}$ in the numerator to match the table entry, so we multiply/divide by that to get:

$$\frac{s+1/2}{(s+1/2)^2+3/4} + \frac{5}{2}\sqrt{\frac{4}{3}}\frac{\sqrt{3/4}}{(s+1/2)^2+3/4}$$

Now we're ready to invert.

We have Table Entry 11 summarized in Table Entries 6 and 7 to save time:

$$f(t) = \mathrm{e}^{-t/2}\cos\left(\sqrt{\frac{3}{4}}\,t\right) + \frac{5}{\sqrt{3}}\mathrm{e}^{-t/2}\sin\left(\sqrt{\frac{3}{4}}\,t\right)$$