Solving two equations in two unknowns.

In this note, we consider how to solve a linear system of two equations and two unknowns x and y.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

Geometrically, we're looking for the intersection of the two lines. Thus, we have exactly one of three possible outcomes:

- Exactly one solution.
- No solution (the lines are parallel).
- An infinite number of solutions (the lines are the same; the solution set *is* the line).

Using Cramer's Rule, if there is exactly one solution, we can write it down using determinants (we used determinants in Calculus III):

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - bc}$$

There are several ways of showing that this is true- We do it in detail in Math 240.

From Cramer's Rule, we also see that there is a fast way of determining if there is not a unique solution: The determinant would be zero- In that case, the two lines have the same slope. One then needs to further check to see if the lines are parallel (no solution) or one line is a multiple of the other.

Examples

- Solve: $\begin{aligned} 3x - 2y &= 1 \\ -x + y &= 2 \end{aligned}$

  SOLUTION:

  $$x = \frac{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}} = \frac{1+4}{3-2} = 5 \qquad y = \frac{\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}} = \frac{6-(-1)}{3-2} = 7$$

- Solve: $\begin{aligned} 4x - 8y &= -4 \\ -2x + 4y &= 2 \end{aligned}$

SOLUTION: The determinant of the coefficient matrix is 0:

$$\begin{vmatrix} 4 & -8 \\ -2 & 4 \end{vmatrix} = 16 - 16 = 0$$

Upon further inspection, we see that multiplying the second equation by 2 would give us the first equation. Thus, there are an infinite numer of solutions that satisfy (either equation, simplified):

$$-x + 2y = -1$$

- Solve: $\begin{aligned} 4x - 8y &= -4 \\ -2x + 4y &= 0 \end{aligned}$

  SOLUTION: The determinant of the coefficient matrix is 0, just as before. However, we notice that the second equation is not a multiple of the first- In this case, there is no solution (parallel lines).

A Special Case: The Homogeneous Equation

Consider the special case of a system of two equations in two unknowns when we set the right hand side to zero. We call this the homogeneous equation.

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

The reason this is special is because in this case, we ALWAYS have one solution, $x = 0$ and $y = 0$. Therefore, we can only have one of TWO possible outcomes: Either the trivial solution (the zero solution) is the only solution, or there is an infinite number of solutions (there is never "no solution").

Therefore, in this case, we can just check the determinant $ad - bc$. If the determinant is zero, there are an infinite number of solutions. Otherwise, zero is the only solution.

Examples:

Solve the following systems:

1. $\begin{aligned} 3x + 2y &= 0 \\ x - 3y &= 0 \end{aligned}$

   In this case, the determinant is $-9 - 2 = -11 \neq 0$. Therefore, zero is the only solution.

2. $\begin{aligned} 2x - y &= 0 \\ 4x - 2y &= 0 \end{aligned}$

   In this case, the determinant is $-4 + 4 = 0$. The lines are the same. The solution is any (x, y) such that $y = 2x$.