

Some Trigonometry and Complexification

The following is a little more explanation of the trig that comes up in Chapter 4.

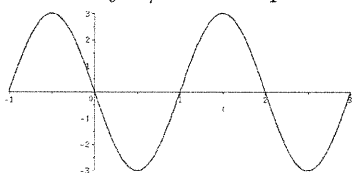
Amplitude, Period, Phase Shift

Given the function $A \cos(\omega t - \delta)$, the amplitude of the function is A , the period is $2\pi/\omega$, the frequency is $\omega/(2\pi)$, the circular frequency is ω , and the phase shift is δ/ω .

So, for example, if we were to graph the function

$$3 \cos\left(\pi t + \frac{\pi}{2}\right),$$

the amplitude is 3, the period is 2, and the phase shift is $-\frac{1}{2}$, which would translate as “shift to the left by 1/4 of the period”. The result:



Cosine Sum Formula

You might recall the cosine sum formula:

$$\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$$

Which in fact is the real part of the complex number:

$$\cos(x - y) + i \sin(x - y) = e^{i(x-y)} = e^{ix} e^{-iy} = (\cos(x) + i \sin(x))(\cos(-y) + i \sin(-y))$$

Continuing the RHS, we get:

$$e^{i(x-y)} = (\cos(x) \cos(y) + \sin(x) \sin(y)) + i(\sin(x) \cos(y) - \sin(y) \cos(x))$$

The real part gives us the cosine difference formula, and the imaginary part gives us the sine difference formula.

Writing a Trig Sum

Using this, we see that:

$$R \cos(\omega t - \delta) = (R \cos(\delta)) \cos(\omega t) + (R \sin(\delta)) \sin(\omega t)$$

This implies that we can write the following sum as a single cosine function:

$$A \cos(\omega t) + B \sin(\omega t) = R \cos(\omega t - \delta) \quad \Leftrightarrow \quad A = R \cos(\delta) \quad B = R \sin(\delta)$$

or, given A, B then:

$$R = \sqrt{A^2 + B^2} \quad \text{and} \quad \delta = \text{Tan}^{-1}\left(\frac{B}{A}\right)$$

(the inverse tangent is the four quadrant version). As a shortcut to remembering this, let

$$z = A + iB$$

then R, δ come from the magnitude and argument of z .

Side Remark: Careful with the parentheses...

If we write the periodic function as $R \cos(\omega(t - \delta))$, then δ itself is the phase shift, but then we don't have the form for the cosine sum formula.

Worked Examples

1. Rewrite the expression as $R \cos(\omega t - \delta)$:

$$\cos(\sqrt{3}t) + 2 \sin(\sqrt{3}t)$$

SOLUTION: Thinking of the complex number $z = 1 + 2i$, then $R = \sqrt{1^2 + 2^2} = \sqrt{5}$, and

$$\delta = \tan^{-1}(2) \approx 1.1 \text{ rad}$$

Therefore,

$$\cos(\sqrt{3}t) + 2 \sin(\sqrt{3}t) = \sqrt{5} \cos(\sqrt{3}t - 1.1)$$

Now it is easy to analyze the sum as a single periodic function- It has an amplitude of $\sqrt{5}$, a period of $2\pi/\sqrt{3}$, and a phase shift of $1.1/\sqrt{3}$ radians.

2. Rewrite the expression as a single cosine:

$$-\cos(3t) + \sin(3t)$$

SOLUTION: Thinking of $z = -1 + i$, the magnitude is $\sqrt{2}$ and the inverse tangent gives $-\pi/4$. BUT, we should notice that our point is actually in the second quadrant, so the argument is actually $\pi - \pi/4 = 3\pi/4$ (we should use the "four quadrant" inverse tangent).

Therefore, the sum is equal to: $\sqrt{2} \cos\left(3t - \frac{3\pi}{4}\right)$.

The amplitude is $\sqrt{2}$, the period is $2\pi/3$, and the phase shift is $\pi/4$.

Exercises

1. Write the expression $3 \cos(5t) - 2 \sin(5t)$ in the form $R \cos(\omega t - \delta)$.
2. Write the expression $-\cos(2t) + \sin(2t)$ in the form $R \cos(\omega t - \delta)$.
3. Write the expression $-\cos(t) - 3 \sin(t)$ in the form $R \cos(\omega t - \delta)$.