Quiz 11 Solutions

1. Write the formula for the Laplace transform of y''' in terms of $Y(s) = \mathcal{L}(y(t))$.
   SOLUTION: $s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$

2. Invert the Laplace transform. You might note that the denominator may be factored.

$$\frac{(5s-12)e^{-3s}}{s^2-5s+6}$$

   SOLUTION: Using partial fractions, we see that

$$\frac{5s-12}{s^2-5s+6} = \frac{3}{s-3} + \frac{2}{s-2}$$

   so that the inverse Laplace transform give us

$$u_3(t)\left(3e^{3(t-3)} + 2e^{2(t-3)}\right)$$

3. Solve, using the Laplace transform:

   (a) $y'' + 4y' + 7y = h(t)$, $y(0) = 3$ and $y'(0) = 0$, where $h(t) = \begin{cases} 1, & \text{if } t < 3 \\ 0, & \text{if } t \geq 3 \end{cases}$
   SOLUTION: See the next page for details.

$$y(t) = 3e^{-2t}\cos(\sqrt{3}\,t) + \frac{6}{\sqrt{3}}e^{-2t}\sin(\sqrt{3}\,t) + u_3(t)f(t-3)$$

   where

$$f(t) = \frac{1}{7} - \frac{1}{7}e^{-2t}\cos(\sqrt{3}\,t) - \frac{2}{7\sqrt{3}}e^{-2t}\sin(\sqrt{3}\,t)$$

   (b) $y'' + 2y' + 5y = \delta_3(t) + u_6(t)$, $y(0) = 1$ and $y'(0) = 2$.
   SOLUTION: See the next page for details.

$$y(t) = e^{-t}\cos(2t) + \frac{3}{2}e^{-t}\sin(2t) + u_2(t)g_1(t-2) + u_6(t)g_2(t-6)$$

   where

$$g_1(t) = \frac{1}{2}e^{-t}\sin(t)$$

   and

$$g_2(t) = \frac{1}{5} - \frac{1}{5}e^{-t}\cos(2t) - \frac{1}{10}e^{-t}\sin(2t)$$

Details for 3(a)

From the other side, we had:

$$Y(s) = \frac{3s+12}{s^2+4s+7} + \frac{\mathrm{e}^{-3s}}{s(s^2+4s+7)}$$

The first term is ready for the inversion process:

$$\frac{3s+12}{s^2+4s+7} = \frac{3(s+2)}{(s+2)^2+3} + \frac{6}{\sqrt{3}}\frac{\sqrt{3}}{(s+2)^2+3}$$

The inverse is $g(t) = 3\mathrm{e}^{-2t}\cos(\sqrt{3}\,t) + \frac{6}{\sqrt{3}}\mathrm{e}^{-2t}\sin(\sqrt{3}\,t)$.

For the second part, we need to perform partial fractions first to get

$$\frac{1}{s(s^2+4s+7)} = \frac{1}{7}\frac{1}{s} - \frac{1}{7}\frac{s+4}{(s+2)^2+3} = \frac{1}{s} - \frac{1}{7}\frac{s+2}{(s+2)^2+3} - \frac{2}{7\sqrt{3}}\frac{\sqrt{3}}{(s+2)^2+3}$$

The inverse of this is straightforward to read off:

$$f(t) = \frac{1}{7} - \frac{1}{7}\mathrm{e}^{-2t}\cos(\sqrt{3}\,t) - \frac{2}{7\sqrt{3}}\mathrm{e}^{-2t}\sin(\sqrt{3}\,t)$$

The inverse is therefore $g(t) + u_3(t)f(t-3)$, where g, f are defined above.

Details for 3(b)

Taking the transform, we have:

$$Y(s) = \frac{\mathrm{e}^{-2s}}{s^2+2s+5} + \frac{\mathrm{e}^{-6s}}{s(s^2+2s+5)} + \frac{s+4}{s^2+2s+5}$$

Taking these each separately, for the first term we see:

$$\frac{1}{s^2+2s+5} = \frac{1}{2}\frac{2}{(s+1)^2+2^2} \quad \to \quad u_2(t)\frac{1}{2}\mathrm{e}^{-(t-2)}\sin(2(t-2))$$

The last term is very similar:

$$\frac{s+4}{s^2+2s+5} = \frac{s+1}{(s+1)^2+2^2} + \frac{3}{2}\frac{2}{(s+1)^2+2^2} \quad \to \quad \mathrm{e}^{-t}\cos(2t) + \frac{3}{2}\mathrm{e}^{-t}\sin(2t)$$

And finally the middle term. Doing partial fractions, we get:

$$\frac{1}{s(s^2+2s+5)} = \frac{1}{5}\frac{1}{s} - \frac{1}{5}\frac{s+2}{s^2+2s+5} = \frac{1}{5}\frac{1}{s} - \frac{1}{5}\frac{s+1}{(s+1)^2+2^2} - \frac{1}{10}\frac{2}{(s+1)^2+2^2}$$

So the inverse of this term will be (including the exponential):

$$u_6(t)\left(\frac{1}{5} - \frac{1}{5}\mathrm{e}^{-(t-6)}\cos(2(t-6)) - \frac{1}{10}\mathrm{e}^{-(t-6)}\sin(2(t-6))\right)$$