Sample Question Solutions (Chapter 3, Math 244)

1. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$
   SOLUTION: False. The characteristic equation is for the homogeneous equation, $r^2 + r + 1 = 0$

   (b) The characteristic equation for $y'' + xy' + e^x y = 0$ is $r^2 + xr + e^x = 0$
   SOLUTION: False. The characteristic equation was defined only for DEs with constant coefficients, since our ansatz depended on constant coefficients.

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.
   SOLUTION: True. It is true generally- If L is a linear operator, then $L(0) = 0$.

   (d) In using the Method of Undetermined Coefficients, the ansatz $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ is equivalent to

   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

   SOLUTION: False- We have to be able to choose the coefficients for each polynomial (for the sine and cosine) independently of each other. In the form:

   $$(Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$$

   the polynomials for the sine and cosine are constant multiples of each other, which may not necessarily hold true. That's why we need one polynomial for the sine, and one for the cosine (so the second guess is the one to use).

   (e) Consider the function:

   $$y(t) = \cos(t) - \sin(t)$$

   Then amplitude is 1, the period is 1 and the phase shift is 0.
   SOLUTION: False. For this question to make sense, we have to first write the function as $R\cos(\omega(t - \delta))$. In this case, the amplitude is R:

   $$R = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

   The period is 2π (the circular frequency, or natural frequency, is 1), and the phase shift δ is:

   $$\tan(\delta) = -1 \quad \Rightarrow \quad \delta = -\frac{\pi}{4}$$

2. Find values of a for which **any** solution to:

   $$y'' + 10y' + ay = 0$$

   will tend to zero (that is, $\lim_{t\to\infty} y(t) = 0$.

SOLUTION: Use the characteristic equation and check the 3 cases (for the discriminant). That is,

$$r^2 + 10r + a = 0 \quad \Rightarrow \quad r = \frac{-10 \pm \sqrt{100 - 4a}}{2}$$

We check some special cases:

- If $100 - 4a = 0$ (or $a = 25$), we get a double root, $r = -5, -5$, or $y_h = \mathrm{e}^{-5t}(C_1 + C_2 t)$, and all solutions tend to zero.
- If the roots are complex, then we can write $r = -5 \pm \beta i$, and we get

  $$y_h = \mathrm{e}^{-5t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

  and again, this will tend to zero for any choice of C_1, C_2.
- In the case that $a < 25$, we have to be a bit careful. While it is true that both roots will be *real*, we also want them to both be *negative* for all solutions to tend to zero.
  - When will they both be negative? If $100 - 4a < 100$ (or $\sqrt{100 - 4a} < 10$). This happens as long as $a > 0$.
  - If $a = 0$, the roots will be $r = -10, 0$, and $y_h = C_1 \mathrm{e}^{-10t} + C_2$- Therefore, I could choose $C_1 = 0$ and $C_2 \neq 0$, and my solution will not go to zero.
  - If $a < 0$, the roots will be mixed in sign (one positive, one negative), so the solutions will not all tend to zero.

CONCLUSION: If $a > 0$, all solutions to the homogeneous will tend to zero.

3. - Compute the Wronskian between $f(x) = \cos(x)$ and $g(x) = 1$.
     SOLUTION: $W(\cos(x), 1) = \sin(x)$
   - Can these be two solutions to a second order linear homogeneous differential equation? Be specific. (Hint: Abel's Theorem)

     SOLUTION: Abel's Theorem tells us that the Wronskian between two solutions to a second order linear homogeneous DE will either be identically zero or never zero on the interval on which the solution(s) are defined.

     Therefore, as long as the interval for the solutions do not contain a multiple of π (for example, $(0, \pi)$, $(\pi, 2\pi)$, etc), then it is possible for the Wronskian for two solutions to be $\sin(x)$.

4. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

SOLUTION: The operator is found by getting all terms in y to one side of the equation, everything else on the other. In this case, we have:

$$L(y) = y' - y^2$$

This is not a linear operator. We can check using the definition:

$$L(cy) = cy' - c^2y^2 \neq cL(y)$$

Furthermore,

$$L(y_1 + y_2) = (y_1' + y_2') - (y_1 + y_2)^2 \neq L(y_1) + L(y_2)$$

5. Find the solution to the initial value problem:

$$u'' + u = 3t + 4 \qquad u(0) = 0 \quad u'(0) = 0$$

SOLUTION: We see that $y_h(t) = C_1 \cos(t) + C_2 \sin(t)$, and $y_p(t) = At + B$ (by Method of Undet Coeffs). Substituting,

$$At + B = 3t + 4 \quad \Rightarrow \quad A = 3, B = 4$$

The solution thus far is $C_1 \cos(t) + C_2 \sin(t) + 3t + 4$. Using the inital conditions,

$$u(0) = 0 \quad \Rightarrow \quad 0 = C_1 + 4 \quad \Rightarrow \quad C_1 = -4$$

$$u'(0) = 0 \quad \Rightarrow \quad 0 = C_2 + 3$$

Therefore,

$$u(t) = -4\cos(t) - 3\sin(t) + 3t + 4$$

6. Solve: $u'' + \omega_0^2 u = F_0 \cos(\omega t), \quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients.

SOLUTION: The characteristic equation is: $r^2 + \omega_0^2 = 0$, or $r = \pm\omega_0 i$. Therefore,

$$u_h = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

Using the Method of Undetermined Coefficients, $u_p = A e^{i\omega t}$. Substitution into the DE:

$$-\omega^2 A e^{i\omega t} + \omega_0^2 A e^{i\omega t} = F_0 e^{i\omega t} \quad \Rightarrow \quad A = \frac{F_0}{\omega_0^2 - \omega^2}$$

This expression is real, so the particular solution is this constant times the cosine. Putting it all together so far, the general solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2} \cos(\omega t)$$

Put in the initial conditions $u(0) = 0$ and $u'(0) = 0$ to see that $C_1 = -\frac{F_0}{\omega_0^2 - \omega^2}$ and $C_2 = 0$ so that

$$u(t) = \frac{F_0(\cos(\omega t) - \cos(\omega_0 t))}{\omega_0^2 - \omega^2}$$

7. Compute the solution to: $u'' + \omega_0^2 u = F_0 \cos(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$ two ways:

- Start over, with Method of Undetermined Coefficients
  SOLUTION: Let $u_p = Ate^{i\omega_0 t}$. Then

$$u_p' = Ae^{i\omega_0 t} + i\omega_0 Ate^{i\omega_0 t} \qquad u_p'' = 2i\omega_0 Ae^{i\omega_0 t} - \omega_0^2 Ate^{i\omega_0 t}$$

  Now,

$$u_p'' + \omega_0^2 u_p = 2i\omega Ae^{i\omega t} = F_0 e^{i\omega_0 t} \quad \Rightarrow \quad A = \frac{F_0}{2i\omega_0} = -\frac{F_0}{2\omega_0} i$$

  We want the real part of $Ate^{i\omega_0 t}$:

$$Ate^{i\omega_0 t} = -\frac{F_0}{2\omega_0} it(\cos(\omega_0 t) + i\sin(\omega_0 t))$$

  which we see is:

$$u_p = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$

  Putting it all together,

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$

  Now, $u(0) = 0$ means that $C_1 = 0$. Differentiating for the second IC,

$$u' = \omega_0 C_2 \cos(\omega_0 t) + \frac{F_0}{2\omega_0} \sin(\omega_0 t) + \frac{F_0 \omega_0}{2\omega_0} t \cos(\omega_0 t)$$

  In this case, $C_2 = 0$ as well, so the particular solution is the full solution.

- Take the limit of $u(t)$ from Question 6 as $\omega \to \omega_0$.
  SOLUTION:

$$\lim_{\omega \to \omega_0} \frac{F_0(\cos(\omega t) - \cos(\omega_0 t))}{\omega_0^2 - \omega^2} = ?$$

  We can use l'Hospital's Rule (differentiate with respect to ω!):

$$= \lim_{\omega \to \omega_0} \frac{-F_0 t \sin(\omega t)}{-2\omega} = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$

8. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

$$t^2 y'' - 2y = 0$$

Find a fundamental set of solutions using Abel's Theorem:

SOLUTION: First, rewrite the differential equation in standard form:

$$y'' - \frac{2}{t^2} y = 0$$

Then $p(t) = 0$ and $W(y_1, y_2) = C\mathrm{e}^0 = C$. On the other hand, the Wronskian is:

$$W(y_1, y_2) = \frac{1}{t}y_2' + \frac{1}{t^2}y_2$$

Put these together:

$$\frac{1}{t}y_2' + \frac{1}{t^2}y_2 = C \quad y_2' + \frac{1}{t}y_2 = Ct$$

The integrating factor is t,

$$(ty_2)' = Ct^2 \quad \Rightarrow \quad ty_2 = C_1t^3 + C_2 \quad \Rightarrow \quad C_1t^2 + \frac{C_2}{t}$$

Notice that we have *both* parts of the homogeneous solution, $y_1 = \frac{1}{t}$ and $y_2 = t^2$.

9. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

   SOLUTION: We can find the differential equation:

$$0.01u'' + 0.05u' + ku = 0 \quad \Rightarrow \quad u'' + 5u' + 100ku = 0$$

   Then the system is *critically damped* if the discriminant (from the quadratic formula) is zero:

$$b^2 - 4ac = 25 - 4 \cdot 100k = 0 \quad \Rightarrow \quad k = \frac{25}{400} = \frac{1}{16}$$

   The system is *underdamped* if the discriminant is negative:

$$25 - 400k < 0 \quad \Rightarrow \quad k > \frac{1}{16}$$

10. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

    (a) $y'' + 4y' + 4y = \mathrm{e}^{-2t}$

    SOLUTUION: We see that

$$y_h(t) = C_1\mathrm{e}^{-2t} + C_2t\mathrm{e}^{-2t}$$

    so that (in putting down our guess, multiply by t^2):

$$y_p = At^2\mathrm{e}^{-2t}$$

    Substituting this into the DE, we should find that $A = 1/2$, so that the full solution is

$$y(t) = C_1\mathrm{e}^{-2t} + C_2t\mathrm{e}^{-2t} + \frac{1}{2}t^2\mathrm{e}^{-2t}$$

(b) $y'' - 2y' + y = te^t + 4$, $y(0) = 1$, $y'(0) = 1$.

With the Method of Undetermined Coefficients, we first get the homogeneous part of the solution,

$$y_h(t) = e^t(C_1 + C_2 t)$$

Now we construct our ansatz (Multiplied by t after comparing to y_h):

$$g_1 = te^t \quad \Rightarrow \quad y_{p_1} = (At + B)e^t \cdot t^2$$

Substitute this into the differential equation to solve for A, B:

$$y_{p_1} = (At^3 + Bt^2)e^t \qquad y'_{p_1} = (At^3 + (3A + B)t^2 + 2Bt)e^t$$

$$y''_{p_1} = (At^3 + (6A + B)t^2 + (6A + 4B)t + 2B)e^t$$

Forming $y''_{p_1} - 2y'_{p_1} + y_{p_1} = te^t$, we should see that $A = \frac{1}{6}$ and $B = 0$, so that $y_{p_1} = \frac{1}{6}t^3e^t$.

The next one is a lot easier! $y_{p_2} = A$, so $A = 4$, and:

$$y(t) = e^t(C_1 + C_2 t) + \frac{1}{6}t^3e^t + 4$$

with $y(0) = 1$, $C_1 = -3$. Solving for C_2 by differentiating should give $C_2 = 4$. The full solution:

$$y(t) = e^t\left(\frac{1}{6}t^3 + 4t - 3\right) + 4$$

(c) $y'' + 4y = 3\sin(2t)$, $y(0) = 2$, $y'(0) = -1$.

The homogeneous solution is $C_1\cos(2t) + C_2\sin(2t)$. For the particular part, consider

$$y'' + 4y = 3(cos(2t) + i\sin(2t)) = 3e^{2it}$$

We'll find the solution to this and take the imaginary part.

$$y_p = Ate^{2it} \qquad y''_p = 4iAe^{2it} - 4Ate^{2it}$$

Therefore,

$$y''_p + 4y_p = 4iAe^{2it} = 3e^{2it} \quad \Rightarrow \quad A = \frac{3}{4i} = \frac{-3i}{4}$$

We want the imaginary part of Ae^{2it}:

$$\text{Imag}\left(\frac{-3i}{4}t(\cos(2t) + i\sin(2t))\right) = -\frac{3t}{4}\cos(2t)$$

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{3}{4}t\cos(2t)$$

With $y(0) = 2$, $c_1 = 2$. Differentiating to solve for c_2, we find that $c_2 = -1/8$.

(d) $y'' + 2y' + y = \cos(2t)$

SOLUTION: The roots to the characteristic equation are $r = -1, -1$. For the particular solution, we replace $\cos(2t)$ by e^{2it} so we can use the complex ansatz:

$$y_p = Ae^{2it} \quad y_p' = 2iAe^{2it} \quad y_p'' = -4Ae^{2it}$$

so that (factoring out Ae^{2it} term):

$$Ae^{2it}(-4 + 2(2i) + 1) = e^{2it} \quad \Rightarrow \quad A = \frac{1}{-3+4i} = \frac{-3-4i}{9+16} = -\frac{3}{25} - \frac{4}{25}i$$

Now we want the real part of Ae^{2it}:

$$Ae^{2it} = \left(-\frac{3}{25} - \frac{4}{25}i\right)(\cos(2t) + i\sin(2t)) =$$

$$\left(-\frac{3}{25}\cos(2t) + \frac{4}{25}\sin(2t)\right) + i\left(-\frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t)\right)$$

We only needed the real part- We included both above so you could see the computations, but in this case, you should note that you could have stopped after the real part of the answer was computed, to get

$$y(t) = e^{-t}(C_1 + C_2 t) - \frac{3}{25}\cos(2t) + \frac{4}{25}\sin(2t)$$

(e) $y'' + 9y = \sum_{m=1}^{N} b_m \cos(m\pi t)$

The homogeneous part of the solution is $C_1\cos(3t) + C_2\sin(3t)$. We see that $3 \neq m\pi$ for $m = 1, 2, 3, \ldots$.

The forcing function is a sum of N functions, the m^{th} function is:

$$g_m(t) = b_m\cos(m\pi t) \quad \Rightarrow \quad y_{p_m} = Ae^{im\pi t}$$

Differentiating,

$$y_{p_m}'' = -m^2\pi^2 Ae^{im\pi t} \quad \Rightarrow \quad y_p'' + 9y_p = (9 - m^2\pi^2)Ae^{im\pi t} = b_m e^{im\pi t}$$

Therefore, $A = b_m/(9 - m^2\pi^2)$, and the full solution is:

$$y(t) = C_1\cos(3t) + C_2\sin(3t) + \sum_{m=1}^{N} \frac{b_m}{9 - m^2\pi^2}\cos(m\pi t)$$

11. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $e^{(3-2i)t}$ (iii) $e^{i\pi}$

NOTE for the SOLUTION: Remember that for any non-negative number A, we can write $A = e^{\ln(A)}$.

- $2^{i-1} = e^{ln(2^{i-1})} = e^{(i-1)\ln(2)} = e^{-\ln(2)}e^{i\ln(2)} = \frac{1}{2}\left(\cos(\ln(2)) + i\sin(\ln(2))\right)$
- $e^{(3-2i)t} = e^{3t}e^{-2ti} = e^{3t}\left(\cos(-2t) + i\sin(-2t)\right) = e^{3t}\left(\cos(2t) - i\sin(2t)\right)$
  (Recall that cosine is an even function, sine is an odd function).
- $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

12. Write $a + ib$ in polar form: (i) $-1-\sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3}-i$

    SOLUTIONS:

    (i) $r = \sqrt{1+3} = 2$, $\theta = -2\pi/3$ (look at its graph, use 30-60-90 triangle):

    $$-1-\sqrt{3}i = 2e^{-\frac{2\pi}{3}i}$$

    (ii) $3i = 3e^{\frac{\pi}{2}i}$

    (iii) $-4 = 4e^{\pi i}$

    (iv) $\sqrt{3} - i = 2e^{-\frac{\pi}{6}i}$

13. Find a second order linear differential equation with constant coefficients whose general solution is given by:

    $$y(t) = C_1 + C_2e^{-t} + \frac{1}{2}t^2 - t$$

    SOLUTION: Work backwards from the characteristic equation to build the homogeneous DE (then figure out the rest):

    The roots to the characteristic equation are $r = 0$ and $r = -1$. The characteristic equation must be $r(r+1) = 0$ (or a constant multiple of that). Therefore, the differential equation is:

    $$y'' + y' = 0$$

    For $y_p = \frac{1}{2}t^2 - t$ to be the particular solution,

    $$y_p'' + y_p' = (1) + (t-1) = t$$

    so the full differential equation must be:

    $$y'' + y' = t$$

14. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

    $$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

    SOLUTION: Write in standard form first-

    $$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

The coefficient functions are all continuous on each of three intervals:

$$(-\infty, 0), (0, 4) \text{ and } (4, \infty)$$

Since the initial time is 3, we choose the middle interval, $(0, 4)$.

15. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.

If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

SOLUTION: This purpose of this question is to see if we can use the properties of linearity to get at the answer.

We see that: $L(3e^{2t}) = -9e^{2t}$, or $L(e^{2t}) = -3e^{2t}$ so:

$$L\left(-\frac{1}{3}e^{2t}\right) = e^{2t}$$

And for the second part,

$$L(t^2 + 3t) = 5t^2 + 3t - 16 \quad \Rightarrow \quad L((-2)(t^2 + 3t)) = -10t^2 + 6t - 32$$

The particular solution is therefore:

$$y_p(t) = -2(t^2 + 3t) - \frac{1}{3}e^{2t}$$

since we have shown that

$$L\left(-2(t^2 + 3t) - \frac{1}{3}e^{2t}\right) = -10t^2 + 6t - 32 + e^{2t}$$

16. Compute the Wronskian of two solutions of the given DE without solving it:

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

Using Abel's Theorem (and writing the DE in standard form first):

$$y'' + \frac{1}{x}y' + \frac{x^2 - \alpha^2}{x^2}y = 0$$

Therefore,

$$W(y_1, y_2) = Ce^{-\int \frac{1}{x}\,dx} = \frac{C}{x}$$

17. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

    SOLUTION: Solving as usual gives:

$$y(t) = \left(\frac{3-\alpha}{5}\right) e^{-2t} + \left(\frac{\alpha+2}{5}\right) e^{3t}$$

    so to make sure the solutions tend to zero, $\alpha = -2$ (to zero out the second term).

18. A mass of 0.5 kg stretches a spring an additional 0.05 meters to get to equilibrium. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

    SOLUTION:

    We use Hooke's Law at equilibrium: $mg - kL = 0$, or

$$k = \frac{mg}{L} = \frac{4.9}{0.05} = 98$$

    For the second part, a stiff spring will not stretch, so L will be small (and k would therefore be large), and a spring that is not stiff will stretch a great deal (so that k will be smaller).

19. A mass of $\frac{1}{2}$ kg is attached to a spring with spring constant 2 (kg/sec^2). The spring is pulled down an additional 1 meter then released. Find the equation of motion if the damping constant is $c = 2$ as well:

    SOLUTION: Just substitute in the values

$$\frac{1}{2}u'' + 2u' + 2u = 0$$

    Pulling down the spring and releasing: $u(0) = 1$, $u'(0) = 0$ (Down is positive)

20. Match the solution curve to its IVP (There is one DE with no graph, and one graph with no DE- You should not try to completely solve each DE).

    (a) $5y'' + y' + 5y = 0$, $y(0) = 10$, $y'(0) = 0$ (Complex roots, solutions go to zero) Graph C

    (b) $y'' + 5y' + y = 0$, $y(0) = 10$, $y'(0) = 0$ (Exponentials, solutions go to zero) Graph D

    (c) $y'' + y' + \frac{5}{4}y = 0$, $y(0) = 10$, $y'(0) = 0$ NOT USED

    (d) $5y'' + 5y = 4\cos(t)$, $y(0) = 0$, $y'(0) = 0$ (Pure Harmonic) Graph B

    (e) $y'' + \frac{1}{2}y' + 2y = 10$, $y(0) = 0$, $y'(0) = 0$ (Complex roots to homogeneous solution, constant particular solution) Graph E

SOLUTION: If the graphs are labeled: Top row: A, B, second row: C, D, and last row E, then the graphs are given above.

21. Be sure you understand the handouts that replaced the homework for 3.7 and 3.8 (all solutions are posted on the class website.)