Solutions- Homework set to replace 7.1

1. Suppose we have two populations, $x(t), y(t)$. Three systems of differential equations are given below.

$$(a)\quad \begin{aligned} x' &= x(1-x) - xy \\ y' &= y(0.75-y) - 0.5xy \end{aligned} \qquad (b)\quad \begin{aligned} x' &= x(1-x) - xy \\ y' &= -0.75y + xy \end{aligned}$$

$$(c)\quad \begin{aligned} x' &= x(1-x) - xy \\ y' &= -0.75y + xy \end{aligned}$$

   One of the models represents a predator-prey system. One of the models represents two species that are cooperating with each other, and one of the models represents two species competing with other.

   Which model could go with which system of DEs? (and explain)

   SOLUTION: Notice that in each system, the first terms are either exponential growth/decay or a logistic equation, followed by an interaction term (xy).

   In system (a), both species decline in the presence of the other (negative coefficient in front of xy), so these species are **competing** for resources. In system (b), we have a **predator and prey**- Population x is the prey, y is the predator. (Ooops- System (c) is the same as (b)- A copy error).

2. Consider the two systems below (nonlinear)

$$\begin{aligned} x' &= 10x(1-0.1\,x) - 20xy \\ y' &= -5y + 0.05xy \end{aligned} \qquad \begin{aligned} x' &= 0.3x - 0.01\,xy \\ y' &= 15y(1-0.05y) + 25xy \end{aligned}$$

   In one of these systems, the prey are very large animals and the predators are very small animals, like elephants and mosquitos. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predators. Determine which system fits this model the best.

   SOLUTION: First, notice that both systems are for predator-prey (opposite signs in the xy term), and in both, x is the prey and y is the predator. The key phrase now is that "each prey is a tremendous benefit for the predators", so that means the second expression ($+25xy$ is the relevant term).

3. Exercise 22, p 363 (Section 7.1)

   Construct the DE's by looking at "Rate In - Rate Out", and make sure your units are matching up. Define $Q_1(t)$ and $Q_2(t)$ to be the ounces of salt in Tanks 1 and 2 (respectively). Before we start, we take note of the rates in and out of each tank- The total amounts of water in tanks 1 and 2 (30 gallons and 20 gallons, respectively) does not change (important for the rates in and out).

Filling in the quantities from the figure on p. 363, we have:

$$\frac{dQ_1}{dt} = \frac{1.5 \text{ gal}}{\text{min}} \cdot \frac{1 \text{ oz}}{\text{gal}} + \frac{1.5 \text{ gal}}{\text{min}} \cdot \frac{Q_2 \text{ oz}}{20 \text{ gal}} - \frac{3 \text{ gal}}{\text{min}} \cdot \frac{Q_1 \text{ oz}}{30 \text{ gal}}$$

$$\frac{dQ_2}{dt} = \frac{1 \text{ gal}}{\text{min}} \cdot \frac{3 \text{ oz}}{\text{gal}} + \frac{3 \text{ gal}}{\text{min}} \cdot \frac{Q_1 \text{ oz}}{30 \text{ gal}} - \frac{4 \text{ gal}}{\text{min}} \cdot \frac{Q_2 \text{ oz}}{20 \text{ gal}}$$

Simplifying, and putting them in order:

$$\frac{dQ_1}{dt} = -\frac{1}{10}Q_1 + \frac{3}{40}Q_2 + \frac{3}{2}$$

$$\frac{dQ_2}{dt} = \frac{1}{10}Q_1 - \frac{1}{5}Q_2 + 3$$

The system is currently non-homogeneous because of the constants 3/2 and 3.

The equilibria are found by solving for where the derivatives are zero. We simplify to make Cramer's Rule easier to apply:

$$\begin{array}{rl} -\frac{1}{10}Q_1 + \frac{3}{40}Q_2 + \frac{3}{2} & = 0 \\ \frac{1}{10}Q_1 - \frac{1}{5}Q_2 + 3 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} -4Q_1 + 3Q_2 & = -60 \\ Q_1 - 2Q_2 & = -30 \end{array}$$

Therefore, using the determinants, the equilibrium solution is:

$$Q_1 = \frac{210}{5} = 42 \qquad Q_2 = \frac{180}{5}$$

The last part of the question shows that a change of variables results in a homogeneous differential equation. That is, if $x_1 = Q_1 - 42$ and $x_2 = Q_2 - 36$, then substitution into the system of DE's:

$$x_1' = Q_1' \qquad x_2' = Q_2'$$

and

$$-\frac{1}{10}(x_1 + 42) + \frac{3}{40}(x_2 + 36) + \frac{3}{2} = -\frac{1}{10}x_1 + \frac{3}{40}x_2$$

Similarly, substitution into the second equation and substituting:

$$\frac{1}{10}(x_1 + 42) - \frac{1}{5}(x_2 + 36) + 3 = \frac{1}{10}x_1 - \frac{1}{5}x_2$$

Notice that our change of coordinates simply shifted the coordinate system so that the equilibrium is now at the origin. To get the initial conditions, make the last substitution:

$$\begin{array}{rl} x_1' & = -\frac{1}{10}x_1 + \frac{3}{40}x_2 \\ x_2' & = \frac{1}{10}x_1 - \frac{1}{5}x_2 \end{array} \qquad x_1(0) = -17, \quad x_2(0) = -21$$

4. Solve the system of equations given by first converting it into a second order linear ODE (then use Chapter 3 methods):

(a) $\begin{aligned} x' &= -2x + y \\ y' &= x - 2y \end{aligned} \quad \Rightarrow \quad y = x' + 2x \quad \Rightarrow \quad (x' + 2x)' = x - 2(x' + 2x)$

So we have: $x'' + 2x' = x - 2x' - 4x$, or

$$x'' + 4x' + 3x = 0 \quad \Rightarrow \quad r^2 + 4r + 3 = 0 \quad \Rightarrow \quad (r + 1)(r + 3) = 0$$

Therefore, $x(t) = C_1 e^{-t} + C_2 e^{-3t}$. For y, we have $y = x' + 2x$, or

$$y = -C_1 e^{-t} - 3C_2 e^{-3t} + 2C_1 e^{-t} + 2C_2 e^{-3t} = C_1 e^{-t} - C_2 e^{-3t}$$

(b) $\begin{aligned} x' &= 2y \\ y' &= -2x \end{aligned} \quad \Rightarrow \quad y = \frac{1}{2}x' \quad \Rightarrow \quad (\frac{1}{2}x')' = -2x$

For $x'' + 4x = 0$, we have $r = \pm 2i$, so $x(t) = C_1 \cos(2t) + C_2 \sin(2t)$, and $y = \frac{1}{2}(-2C_1 \sin(2t) + 2C_2 \cos(2t)) = -C_1 \sin(2t) + C_2 \cos(2t)$

5. Convert the following second order differential equations into a system of autonomous, first order equations. Using methods from Chapter 3, give the solution to the system.

(a) $y'' + 4y' + 3y = 0$

SOLUTION: Let $x_1 = y$, $x_2 = y'$. Then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -3x_1 - 4x_2 \end{aligned}$$

From the characteristic equation, $(r + 1)(r + 3) = 0$, so

$$x_1(t) = C_1 e^{-t} + C_2 e^{-3t}, \qquad x_2(t) = x_1'(t) = -C_1 e^{-t} - 3C_2 e^{-3t}$$

(b) $y'' + 5y' = 0$. Repeating the same steps as last time,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -5x_2 \end{aligned}$$

Now, using the characteristic equation, $r^2 + 5r = 0$ so $r = 0, -5$.

$$x_1(t) = C_1 + C_2 e^{-5t} \quad \text{and} \quad x_2(t) = -5C_2 e^{-5t}$$

(c) $y'' + 4y = 0$

SOLUTION:

We see that $y(t) = C_1 \cos(2t) + C_2 \sin(2t)$, and the system (let $x_1 = y, x_2 = y'$) is:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1(t) &= C_1 \cos(2t) + C_2 \sin(2t) \\ x_2(t) &= -2C_1 \sin(2t) + 2C_2 \cos(2t) \end{aligned}$$

(d) $y'' - 2y' + y = 0$
SOLUTION:

$$\begin{array}{ll} x_1' &= x_2 \\ x_2' &= -x_1 + 2x_2 \end{array} \quad \Rightarrow \quad \begin{array}{r} x_1(t) = \mathrm{e}^t(C_1 + C_2 t) \\ x_2(t) = \mathrm{e}^t(C_1 + C_2 t) + \mathrm{e}^t(0 + C_2) = \mathrm{e}^t(C_1 + C_2 + C_2 t) \end{array}$$

6. Give the solution to each system of equations. If it has an infinite number of solutions, give your answer in vector form:

$$\begin{aligned} 3x + 2y &= 1 \\ 2x - y &= 3 \end{aligned}$$

We have a unique solution:

$$x = \frac{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = 1 \qquad y = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = -1$$

$$\begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= 3 \end{aligned}$$

SOLUTION: This system has no solution- parallel lines (no point of intersection).

$$\begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= 2 \end{aligned}$$

SOLUTION: This system has an infinite number of solutions- Any point on either line would do. Writing the equation of the line in vector form (through the point (1,2) in the direction of $(2, -3)$ since the slope is $-3/2$):

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$