Exercise Set 3 (HW to replace 7.3, 7.5)

This homework is all about solving for eigenvalues and eigenvectors, and we'll also do some visualization and classification of equilibria.

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

2. For each matrix, find the eigenvalues and eigenvectors. Note that they may be complex (when solving the quadratic equation).

   (a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (f) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

3. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3, if A is given below:

   (a) $A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

4. Consider the expression: $\mathrm{e}^{\lambda t}\mathbf{v}$, where $\mathbf{v}$ is a two dimensional non-zero vector that we'll assume is fixed. Below we want to consider what happens graphically as we change λ.

   - If $\lambda < 0$, what happens specifically as $t \to \infty$? What happens as $t \to -\infty$?
   - If $\lambda = 0$, what happens specifically as $t \to \infty$? What happens as $t \to -\infty$?
   - If $\lambda > 0$, what happens specifically as $t \to \infty$? What happens as $t \to -\infty$?

5. Give the general solution to each system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, and sketch a phase plane (solutions in the x_1, x_2 plane). Identify the origin as a *sink*, *source* or *saddle*:

   (a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

   (d) $A = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}$