Summary- Elements of Chapters 7 and 9

We started with some basic matrix algebra- Be sure you know how to perform matrix-vector multiplication and matrix-matrix multiplication for 2×2 matrices.

Eigenvalues and Eigenvectors

For the following, we are solving the system:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}$$

1. Definition: If there is a constant λ and a non-zero vector $\mathbf{v}$ that solves

$$\begin{array}{lll} av_1 & +bv_2 & = \lambda v_1 \\ cv_1 & +dv_2 & = \lambda v_2 \end{array}$$

   then λ is an eigenvalue, and $\mathbf{v}$ is an associated eigenvector.

2. To solve for the eigenvalues, note the logical progression:

$$\begin{array}{lll} av_1 & +bv_2 & = \lambda v_1 \\ cv_1 & +dv_2 & = \lambda v_2 \end{array} \Leftrightarrow \begin{array}{rrr} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array} \tag{1}$$

   This system has a non-zero solution for v_1, v_2 only if the two lines are multiples of each other. In that case, the determinant must be zero.

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad \Rightarrow \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

   And this is the **characteristic equation**. This is formallly solved via the quadratic formula, but we would typically factor it or complete the square. For each λ, we must go back and solve Equation (1) to find $\mathbf{v}$. For example, if we have the line on the left, the eigenvector can be written down directly (as long as the equation is not $0 = 0$)

$$(a-\lambda)v_1 + cv_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -c \\ a-\lambda \end{bmatrix}$$

Solve $\mathbf{x}' = A\mathbf{x}$

1. We make the ansatz: $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$, substitute into the DE, and we find that $\lambda, \mathbf{v}$ must be an eigenvalue, eigenvector of the matrix A.

2. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

   The solution is one of three cases, depending on Δ:

   - Real λ_1, λ_2 with two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

   - Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1\mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2\mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector (which is not needed). Determine $\mathbf{w}$, where:

$$\begin{aligned} (a-\lambda)x_0 + cy_0 &= w_1 \\ cx_0 + (d-\lambda)y_0 &= w_2 \end{aligned}$$

  Then
$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = \mathrm{e}^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$

  Note: In this solution, we don't have arbitrary constants- rather, we're writing the solution in terms of the initial conditions.

You might find this helpful- Below there is a chart comparing the solutions from Chapter 3 to the solutions in Chapter 7:

	Chapter 3	Chapter 7
Form:	$ay'' + by' + cy = 0$	$\mathbf{x}' = A\mathbf{x}$
Ansatz:	$y = \mathrm{e}^{rt}$	$\mathbf{x} = \mathrm{e}^{\lambda t}\mathbf{v}$
Char Eqn:	$ar^2 + br + c = 0$	$\det(A - \lambda I) = 0$
Real Solns	$y = C_1\mathrm{e}^{r_1 t} + C_2\mathrm{e}^{r_2 t}$	$\mathbf{x}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$
Complex	$y = C_1\mathrm{Re}(\mathrm{e}^{rt}) + C_2\mathrm{Im}(\mathrm{e}^{rt})$	$\mathbf{x}(t) = C_1\mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2\mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$
SingleRoot	$y = \mathrm{e}^{rt}(C_1 + C_2 t)$	$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(\mathbf{x}_0 + t\mathbf{w}\right)$

Classification of the Equilibria

The origin is always an equilibrium solution to $\mathbf{x}' = A\mathbf{x}$, and we can use the Poincaré Diagram to help us classify the origin (in Chapter 7) or other equilibrium solutions (in Chapter 9).

Solve General Nonlinear Equations

We don't have a method that will work on every system of nonlinear differential equations, although there are some tricks we can try with special cases- that is, given the system

$$\begin{aligned} \tfrac{dx}{dt} &= f(x,y) \\ \tfrac{dy}{dt} &= g(x,y) \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

And we might get lucky if it is in the form of an equation from Chapter 2.

Local Analysis of Nonlinear Equations

Often, we can perform a local analysis of a system of nonlinear DEs by "linearizing about the equilibria". Given

$$\begin{aligned} \tfrac{dx}{dt} &= f(x,y) \\ \tfrac{dy}{dt} &= g(x,y) \end{aligned}$$

- Find the equilibrium solutions ($f(x,y) = 0$ and $g(x,y) = 0$).
- At each equilibrium, we perform the local analysis by first linearizing, then we classify the equilibrium. Given an equilibrium at $x = a, y = b$, we construct the matrix (the Jacobian) at that point:

$$\begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix}$$

  Use the Poincaré Diagram to classify the equilibrium.

Modeling

Recall that we also did some modeling in these sections- Primarily, we looked at the predator-prey model and the tank mixing problem (with multiple tanks). Given a system that represents two populations, you should be able to determine if the system represents predator-prey, competing species, or cooperating species.