Complex Integrals and the Laplace Transform

There are a few computations for which the complex exponential is very nice to use. Before we get too much farther, here are some facts about integrating expressions that involve i:

Theorem: $\int \mathrm{e}^{(bi)t}\,dt = \frac{1}{bi}\mathrm{e}^{(bi)t}$
Proof: To prove this, we use Euler's Formula to put the new integral in terms of the usual real integral:

$$\int \mathrm{e}^{(bi)t}\,dt == \int \cos(bt) + i\sin(bt)\,dt = \int \cos(bt)\,dt + i\int \sin(bt)\,dt =$$

$$\frac{1}{b}\sin(bt) - \frac{i}{b}\cos(bt) = \frac{\sin(bt) - i\cos(bt)}{b}$$

And

$$\frac{1}{bi}\mathrm{e}^{(bt)i} = \frac{\cos(bt) + i\sin(bt)}{bi} \cdot \frac{i}{i} = \frac{-\sin(bt) + i\cos(bt)}{-b} = \frac{\sin(bt) - i\cos(bt)}{b}$$

Therefore, these quantities are the same.

Theorem: $\int \mathrm{e}^{(a+bi)t}\,dt = \frac{1}{(a+bi)}\mathrm{e}^{(a+bi)t}$
You can work this out, but it is more complicated since we'll need to do integration by parts twice for each integral. It is a nice exercise to try out when you have a little time.

Theorem: The main computational technique is using the following:

$$\int \mathrm{e}^{at}\cos(bt)\,dt = \mathrm{Re}\left(\int \mathrm{e}^{(a+bi)t}\,dt\right) = \mathrm{Re}\left(\frac{1}{a+ib}\mathrm{e}^{(a+ib)t}\right)$$

$$\int \mathrm{e}^{at}\sin(bt)\,dt = \mathrm{Im}\left(\int \mathrm{e}^{(a+bi)t}\,dt\right) = \mathrm{Im}\left(\frac{1}{a+ib}\mathrm{e}^{(a+ib)t}\right)$$

Worked Example:

1. Use complex exponentials to compute $\int \mathrm{e}^{2t}\cos(3t)\,dt$.

   SOLUTION: We note that $\mathrm{e}^{2t}\cos(3t) = \mathrm{Re}(\mathrm{e}^{(2+3i)t})$, so:

$$\int \mathrm{e}^{2t}\cos(3t)\,dt = \mathrm{Re}\left(\frac{1}{2+3i}\mathrm{e}^{(2+3i)t}\right)$$

   Simplifying the term inside the parentheses and multiplying out the complex terms:

$$\mathrm{e}^{2t}\left(\frac{2-3i}{4+9}\right)(\cos(3t) + i\sin(3t)) =$$

$$\mathrm{e}^{2t}\left[\left(\frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t)\right) + i\left(-\frac{3}{13}\cos(3t) + \frac{2}{13}\sin(3t)\right)\right]$$

Therefore,

$$\int e^{2t}\cos(3t)\,dt = e^{2t}\left(\frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t)\right)$$

In fact, we get the other integral for free:

$$\int e^{2t}\sin(3t)\,dt = e^{2t}\left(\frac{-3}{13}\cos(3t) + \frac{2}{13}\sin(3t)\right)$$

2. Use complex exponentials to compute $\int \sin(at)\,dt$

   This one is simple enough to do without using complex exponentials, but it does still work.

$$\int \sin(at)\,dt = \mathrm{Im}\left(\left(\int e^{(at)i}\,dt\right)\right) = \mathrm{Im}\left(\frac{1}{ai}(\cos(at) + i\sin(at)\right) =$$

$$\mathrm{Im}\left(\frac{-i}{a}(\cos(at) + i\sin(at))\right) = \mathrm{Im}\left(\frac{1}{a}\sin(at) + i\left(\frac{-1}{a}\cos(at)\right)\right) = \frac{-1}{a}\cos(at)$$

3. Use complex exponentials to compute the Laplace transform of $\cos(at)$:

   SOLUTION: Note that $\cos(at) = \mathrm{Re}(e^{(at)i})$

$$\mathcal{L}(\cos(at)) = \int_0^\infty e^{-st}\cos(at)\,dt = \mathrm{Re}\left(\left(\int_0^\infty e^{-st}e^{(ai)t}\,dt\right)\right) =$$

$$\mathrm{Re}\left(\left(\int_0^\infty e^{-(s-ai)t}\,dt\right)\right) = \mathrm{Re}\left(\frac{-1}{(s-ai)}e^{-(s-ai)t}\Big|_{t=0}^{t\to\infty}\right.$$

   What happens to our expression as $t \to \infty$? The easiest way to take the limit is to check the magnitude (see if it is going to zero):

$$\left|\frac{-1}{s-ai}e^{-st}e^{(ai)t}\right| = \left|\frac{-1}{s-ai}\right| \cdot \left|e^{-st}\right| \cdot \left|e^{(ai)t}\right|$$

   Now, the first term is a constant and $e^{(at)i}$ is a point on the unit circle (so its magnitude is 1). Therefore, the magnitude depends solely on e^{-st}, where s is any real number.

   And, the function $e^{-st} \to 0$ as $t \to \infty$ for any $s > 0$. Therefore,

$$\lim_{t\to\infty} \frac{-1}{(s-ai)}e^{-(s-ai)t} = 0$$

   and the Laplace transform is:

$$\mathcal{L}(\cos(at)) = \mathrm{Re}\left(0 - \frac{-1}{s-ai}\right) = \mathrm{Re}\left(\frac{s+ai}{s^2+a^2}\right) = \frac{s}{s^2+a^2}$$

   As a side remark, we get the Laplace transform of $\sin(at)$ for free since it is the imaginary part.

Homework Addition to Section 6.1

1. Use complex exponentials to compute $\int e^{-2t} \sin(3t)\, dt$.

2. Use complex exponentials to compute the Laplace transform of $\sin(at)$.

3. Use complex exponentials to compute the Laplace transform of $e^{at} \sin(bt)$ and $e^{at} \cos(bt)$ (compare to exercises 13, 14).

4. Prove that e^t goes to infinity faster than any polynomial. You can do that by showing

$$\lim_{t \to \infty} \frac{t^n}{e^t} = 0$$

5. We can show that $f(x) < g(x)$ for all $x \geq a$ by proving two things: (i) $f(a) < g(a)$, and (ii) $f'(x) < g'(x)$ for all $x > a$. Use this idea to prove that $\ln(t) < t$ for all $t \geq 1$ (it is true for all $t > 0$, but we wouldn't be able to use this argument for $0 < t < 1$).

6. Show that, if $f(t)$ is bounded (that is, there is a constant A so that $|f(t)| \leq A$ for all t), then f is of exponential order (do this by finding K, a and M from the definition).

7. If the function is of exponential order, find the K, a and M from the definition. Otherwise, state that it is not of exponential order.

   Something that may be handy from algebra: $A = e^{\ln(A)}$.

   (a) $\sin(t)$
   (b) $\tan(t)$
   (c) t^3
   (d) e^{t^2}
   (e) 5^t
   (f) t^t

8. Use complex exponentials to find the Laplace transform of $t \sin(at)$.