Graphical Analysis (Phase Portraits)

Given $\mathbf{x}' = A\mathbf{x}$, then our ansatz was $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$. Assuming λ to be real, then geometrically in the (x_1, x_2) plane, letting $-\infty < t < \infty$, the graph is a ray emanating from the origin in the direction of $\mathbf{v}$.

- We note that if $\lambda > 0$ and $t \in [0, \infty)$, then the ray starts at the tip of $\mathbf{v}$, then moves away from the origin. If $\lambda < 0$, then $\mathbf{x}(t)$ would move towards the origin.

  There are three choices for classifying the origin in the case that we have real values of λ:

  - If λ_1, λ_2 have different signs the origin is a SADDLE.
  - If λ_1, λ_2 are both negative, the origin is a SINK.
  - If λ_1, λ_2, are both positive, the origin is a SOURCE.
  - If $\lambda_1 = \lambda_2$, then we have either a degenerate source or a degenerate sink.

- If $\lambda = \alpha + \beta i$, then
$$\mathrm{e}^{(\alpha+\beta i)t} = \mathrm{e}^{\alpha t}\mathrm{e}^{i(\beta t)} = \mathrm{e}^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$
  Therefore, the value of α will tell us if our solution is growing larger ($\alpha > 0$) or growing smaller ($\alpha < 0$), and the β defines the rotation. Recall that:

  - If $\alpha < 0$, the origin is a spiral sink.
  - If $\alpha = 0$, the origin is a center.
  - If $\alpha > 0$, the origin is a spiral source.

  (These can be verified using Poincare classification).

Saddles

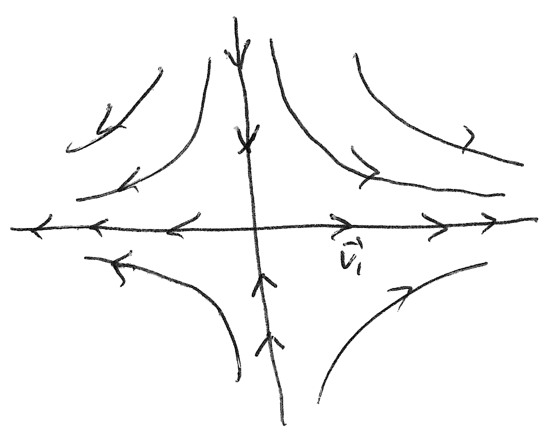
Suppose our linear system has eigenvalues and eigenvectors that are given by:

$$\lambda_1 = 2, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda_2 = -3, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To graph this is the phase plane, first draw in the lines that are in the direction of vectors $\mathbf{v}_1, \mathbf{v_2}$. Along the line with $\mathbf{v}_1$, the eigenvalue is positive, so in time, the solution $\mathbf{x}(t)$ will move away from the origin. Indicate that with arrows away from the origin.

Similarly, along the line with $\mathbf{v}_2$, the eigenvalue is negative, so solutions that start on this line move towards the origin (indicate so with arrows).

Finally, all solutions will be going to infinity along the vector with the positive eigenvalue, so you can sketch those (roughly is OK) as well. See the graph below.



Algebraically, notice that if

$$\mathbf{x}(t) = C_1 e^{-t}\mathbf{v}_1 + C_2 e^{2t}\mathbf{v}_2$$

where C_1, C_2 are not zero, then over time, the first term (with e^{-t}) will go to zero, and the second term, (the one with e^{2t}) will dominate. That what we're seeing in the graph.

The technique we outlined here works for all saddles.

Sinks

Consider now the case where both eigenvalues are negative For example,

$$\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = -2, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

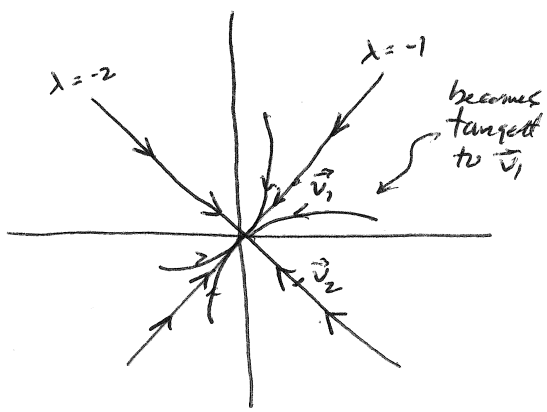
We start exactly like last time, by drawing the two lines in the directions of our two eigenvectors. In this case, all arrows will point towards the origin.

Here's a key bit of information: Consider the full solution below where we take C_1, C_2 to be random (not zero).

$$\mathbf{x}(t) = C_1 e^{-t}\mathbf{v}_1 + C_2 e^{-2t}\mathbf{v}_2$$

Then the second term, the one with the exponent MOST negative, will drop to zero much more rapidly than the first term.

Therefore, solutions moving to the origin will be **tangent** to $\mathbf{v}_1$. That's what you see- A bending of other solution curves towards $\mathbf{v}_1$.



Key summary: In a sink, look at the eigenvalues (which will both be negative). The one that is **less negative** corresponds to the vector to which other solutions become tangent as $t \to \infty$.

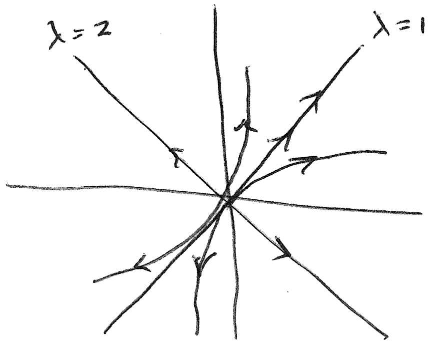
Sources

A source can be changed into a sink by running time in reverse. That means that the graph of a source can be drawn by thinking about the eigenvalues as negative instead of positive, but then also reversing the arrows.

For example, the source in the case that we have:

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = 2, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

looks the same (the curves are the same) as the corresponding sink (if we negate both eigenvalues), but the arrows just point in the opposite direction.



Centers and Spirals

It is generally hard to determine how "tight" to draw a spiral, or how elongated to draw an ellipse. For these, you can be pretty rough. The one thing to watch for in a spiral and center is whether the solution is moving clockwise or counterclockwise.

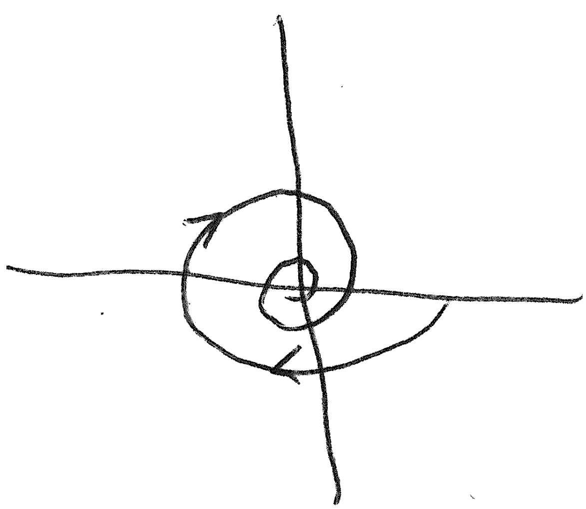
Example: Let the matrix A (in $\mathbf{x}' = A\mathbf{x}$) be given by:

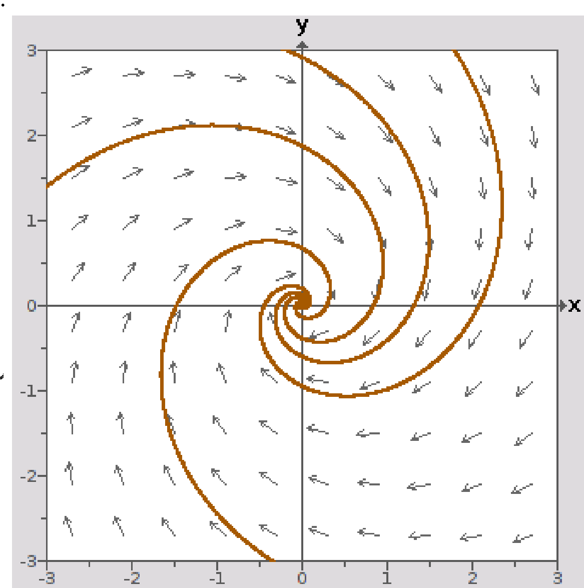
$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rl} \mathrm{Tr}(A) & = -2 \\ \det(A) & = 5 \\ \lambda & = -1 \pm 2i \end{array}$$

So we have a spiral sink, and we want to know the direction of motion. Since $\mathbf{x}' = A\mathbf{x}$, then we can put in a convenient vector $\mathbf{x}$ from which to compute the direction. I like to use $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, but any vector will work. In that case,

$$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

(Yes, it's always the first column of the matrix- That's a shortcut you can use if you want) Starting from $(1, 0)$, we move horizontally by -1 and vertically by -2, which means the motion is **clockwise**. The rough sketch (and a good computer sketch) are both below.





Example 2: Let the matrix A (in $\mathbf{x}' = A\mathbf{x}$) be given by:

$$A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \operatorname{Tr}(A) &= 0 \\ \det(A) &= 9 \\ \lambda &= \pm 3i \end{aligned}$$

So we have a center, and we want to know the direction of motion. Just like last time, we use a convenient vector (same as last time) to compute the direction:

$$\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Starting from $(1, 0)$, we move horizontally by $+1$ and vertically by -5, which means the motion is **clockwise** again. Below is a rough sketch, together with a computer rendering.