Periodic Forcing and Complex Functions

These notes describe how we can use the complex exponential function to compute the particular solution to:

$$ay'' + by' + cy = F(t)$$

when the forcing function F is periodic- or, in particular, when it is a sine or cosine function.

For example, suppose we have:

$$y'' - y' - 2y = \cos(t)$$

We could solve for y_p by assuming that $y_p = A\cos(t) + B\sin(t)$, but we can "embed" the problem into a slightly more general problem:

$$y'' - y' - 2y = \cos(t) + i\sin(t) = \mathrm{e}^{it}$$

We can now solve for y_p by assuming that $y_p = A\mathrm{e}^{it}$- A lot easier! We just have to interpret the answer correctly. We see that if $y_p = u(t) + iv(t)$ and solves

$$y'' - y' - 2y = \cos(t) + i\sin(t)$$

then $u(t)$ will solve the problem if just $\cos(t)$ is on the RHS. Similarly, if the original problem had $\sin(t)$ on the RHS, then we would have used $v(t)$ for y_p for the original problem. Let's check this out more closely.

If we substitute as mentioned, then

$$y_p = A\mathrm{e}^{it} \qquad y_p' = Ai\mathrm{e}^{it} \qquad y_p'' = -A\mathrm{e}^{it}$$

Therefore, substituting these into the DE and factoring on the LHS:

$$A\mathrm{e}^{it}(-1 - i - 2) = \mathrm{e}^{it} \quad \Rightarrow \quad A = \frac{1}{-3-i} = -\frac{1}{3+i}$$

Remember that we multiply the fraction top and bottom by the conjugate to make the fraction $a + ib$ form:

$$A = \frac{-1}{3+i}\,\frac{3-i}{3-i} = \frac{-(3-i)}{10}$$

To compute y_p, multiply $A\mathrm{e}^{it}$ out. You can FOIL it all out, or you can recall that we only need the real part-

$$y_p = A\mathrm{e}^{it} = \left(-\frac{3}{10} + \frac{1}{10}i\right)(\cos(t) + i\sin(t)) = \left(-\frac{3}{10}\cos(t) - \frac{1}{10}\sin(t)\right) + i\left(\frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)\right)$$

The real part of y_p is our solution:

$$\mathrm{Real}(y_p(t)) = -\frac{3}{10}\cos(t) - \frac{1}{10}\sin(t)$$

Of course, the full solution to the original ODE would be:

$$y(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{2t} - \frac{3}{10}\cos(t) - \frac{1}{10}\sin(t)$$

If the original problem had been:

$$y'' - y' - 2y = \sin(t)$$

then the solution would have used the imaginary part of y_p in our previous computation, and the solution would be:

$$y(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{2t} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)$$

Summary of the technique

Given

$$ay'' + by' + cy = \cos(\omega t)$$

we can use $y_p = A\mathrm{e}^{i\omega t}$ as our ansatz (assuming that $\cos(\omega t)$ does not solve the homogeneous equation). Substituting this guess in,

$$y_p = A\mathrm{e}^{i\omega t}, \quad y_p' = Ai\omega\mathrm{e}^{i\omega t}, \quad y_p'' = -A\omega^2\mathrm{e}^{i\omega t}$$

we get:

$$A\mathrm{e}^{i\omega t}(-a\omega^2 + ib\omega + c) = \mathrm{e}^{i\omega t}$$

so that

$$A = \frac{1}{c - a\omega^2 + ib\omega}$$

We then compute $A\mathrm{e}^{i\omega t}$ and keep only the real part for y_p of the original DE.

Example

$$y'' - 4y' + 3y = \cos(2t)$$

Initially, we note that $r^2 - 4r + 3 = 0$ gives $r = 1, 3$. We then find the particular solution by taking the ansatz as $y_p = A\mathrm{e}^{2it}$. Differentiating twice and substituting it into the DE (and factoring the LHS), we get

$$A\mathrm{e}^{2it}(-4 - 4(2i) + 3) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-1-8i} = \frac{-1}{1+8i} = \frac{-(1-8i)}{65}$$

Now we compute $A\mathrm{e}^{2it}$ and keep the real part.

$$A\mathrm{e}^{2it} = \left(-\frac{1}{65} + \frac{8}{65}i\right)(\cos(2t) + i\sin(2t))$$

The real part is straightforward to compute, and the full solution is given by

$$y(t) = C_1\mathrm{e}^{t} + C_2\mathrm{e}^{3t} - \frac{1}{65}\cos(2t) - \frac{8}{65}\sin(2t)$$

Practice Problems

Solve each DE below using the complex function technique described here. Be sure to give the full solution for each.

1. $y'' + 2y' + y = \cos(2t)$
2. $y'' + 2y' + y = \sin(2t)$
3. $y'' + 2y' + y = 3\cos(2t)$ (Hint: Use the answer to the first problem)
4. $y'' + 3y' + 2y = \cos(t)$
5. $y'' + y = \cos(2t)$
6. $y'' + y = \sin(t)$ (Careful with this one!)

Solutions

1. $\mathrm{e}^{-t}(C_1 + C_2 t) - \frac{3}{25}\cos(2t) + \frac{4}{25}\sin(2t)$
2. $\mathrm{e}^{-t}(C_1 + C_2 t) - \frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t)$
3. Multiply the particular solution in #1 by 3:
   $\mathrm{e}^{-t}(C_1 + C_2 t) - \frac{9}{25}\cos(2t) + \frac{12}{25}\sin(2t)$
4. $C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-2t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$
5. $C_1\cos(t) + C_2\sin(t) - \frac{1}{3}\cos(2t)$
6. Multiply your ansatz by t so that

   $$y_p = At\mathrm{e}^{it} \qquad y_p' = A\mathrm{e}^{it}(1 + it) \qquad y_p'' = A\mathrm{e}^{it}(2i - t)$$

   Substitute, and get

   $$A\mathrm{e}^{it}(2i) = \mathrm{e}^{it} \quad \Rightarrow \quad A = \frac{1}{2i}$$

   And then find that

   $$y(t) = C_1\cos(t) + C_2\sin(t) - \frac{1}{2}\,t\,\cos(t)$$

Complex Exponential and Integration

As one last example, suppose we wanted to compute

$$\int e^{2x}\sin(3x)\,dx$$

We would have to use integration by parts twice, then get the same integral on both sides of the equation. Doing this with $u=\sin(3x)$ and $dv=e^{2x}$, we get

$$\int e^{2x}\sin(3x)\,dx=\frac{1}{2}e^{2x}\sin(3x)-\frac{3}{4}e^{2x}\cos(3x)-\frac{9}{4}\int e^{2x}\sin(3x)\,dx$$

Bring the integrals together and solve,

$$\int e^{2x}\sin(3x)\,dx=\frac{2}{13}e^{2x}\sin(3x)-\frac{3}{13}e^{2x}\cos(3x)$$

Rather than doing that, we can instead compute the "larger" integral:

$$\int e^{2x}(\cos(3x)+i\sin(3x))\,dx=\int e^{(2+3i)x}\,dx$$

and then take the imaginary part (since we want the integral dealing with sine). The integral of the complex exponential is especially easy:

$$\frac{1}{2+3i}e^{(2+3i)x}$$

Expand this out and get the imaginary part:

$$e^{2x}\left(\frac{2}{13}-\frac{3}{13}i\right)(\cos(3x)+i\sin(3x))$$

And indeed the imaginary part is:

$$e^{2x}\left(-\frac{3}{13}\cos(3x)+\frac{2}{13}\sin(3x)\right)$$

which is what we got using the traditional method.

Extra Practice

Try these integrals (for the solutions, type them into Maple!)

1. $\int e^{x}\cos(x)\,dx$
2. $\int e^{-2x}\sin(x)\,dx$
3. $\int e^{3x}\cos(2x)\,dx$