4(a): $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

HW- Graphical analysis and Poincare Classification

1. The matrices below represent the matrix A in a linear system, $\mathbf{x}' = A\mathbf{x}$. Classify the equilibrium (the origin) using the Poincaré Classification:

   (a) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$

   (b) $\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

2. Discuss how the classification of the origin changes with α, given the matrix below.

   (a) $\begin{bmatrix} \alpha & -1 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} \alpha & \alpha \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} \alpha & 1 \\ \alpha & \alpha \end{bmatrix}$

3. Suppose we are given $\mathbf{x}' = A\mathbf{x}$, and we compute the eigenvalues and eigenvectors (below). Draw a sketch of the phase plane in each case.

   (a) $\lambda_1 = -1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (b) $\lambda_1 = -1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_2 = -2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (c) $\lambda_1 = 1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

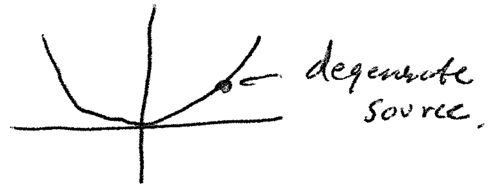
4. Each matrix below represents the matrix A in the system $\mathbf{x}' = A\mathbf{x}$. For each, (i) first determine the eigenvalues/ eigenvectors (if needed), (ii) write down the general solution, and (iii) draw a sketch of the phase plane (the (x_1, x_2) plane).

   (a) $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

1(a) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ $Tr(A) = 4$

$det(A) = 3 - -1 = 4$

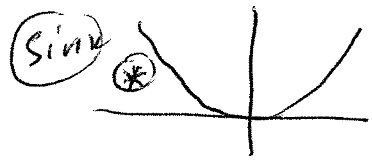
$\Delta = 4^2 - 4 \cdot 4 = 0$



~~(b)~~ (C) $\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$ $Tr(A) = -5/4$

$det(A) = 1/4$

$\Delta = \frac{25}{16} - 4 \cdot \frac{1}{4} = \frac{25-16}{16} = \frac{9}{16}$

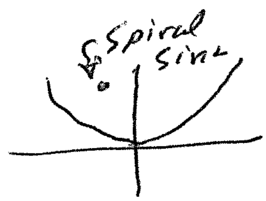


Sorry!

(b) ~~c~~ $\begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix}$ $Tr(A) = -1$

$det(A) = \frac{1}{4} - -1 = 5/4$

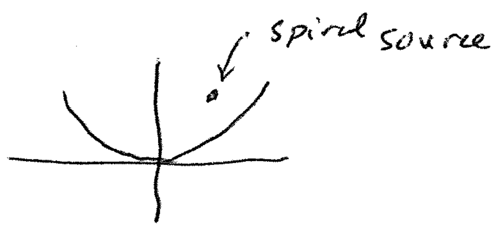
$\Delta = 1 - 4(5/4) = -4$



d) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ $Tr(A) = 2$

$det(A) = -3 + 8 = 5$

$\Delta = 2^2 - 4(5) = -16$

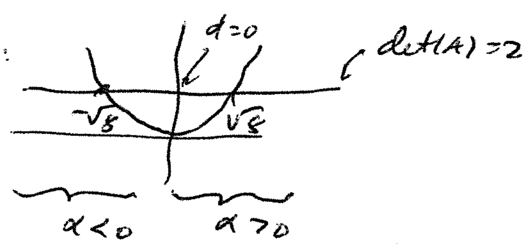


2. (a) $\begin{bmatrix} \alpha & -1 \\ 2 & 0 \end{bmatrix}$ $Tr(A) = \alpha$

$det(A) = 2$

$\Delta = \alpha^2 - 4(2) = \alpha^2 - 8$

Note: If det (A) is fixed and positive, then we move along the (Tr, det) plane:



If $\alpha < -\sqrt{8}$, ~~spiral~~ sink

$\alpha > \sqrt{8}$, degen sink

$-\sqrt{8} < \alpha < 0$, spiral sink

$\alpha = 0$ Center

$0 < \alpha < \sqrt{8}$ spiral source

$\alpha = \sqrt{8}$ degen source

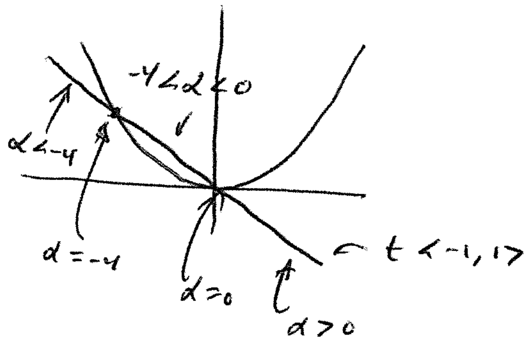
$\alpha > \sqrt{8}$ Source

2 b) $\begin{bmatrix} \alpha & \alpha \\ 1 & 0 \end{bmatrix}$ $Tr(A) = \alpha$
$\det(A) = -\alpha$
$\Delta = \alpha^2 + 4\alpha$

If we think of $Tr(A)$ as x, det as y, and α as like t, then the curve in parametric form is $\langle t, -t \rangle = t\langle -1, 1 \rangle$

Sign analysis:

	$\alpha = -4$		$\alpha = 0$	
$Tr(A) = \alpha$	$-$		$-$		$+$
$\det(A) = -\alpha$	$+$		$+$		$-$
$\Delta = \alpha^2 + 4\alpha$	$+$		$-$		$+$
	sink		spiral sink		saddle



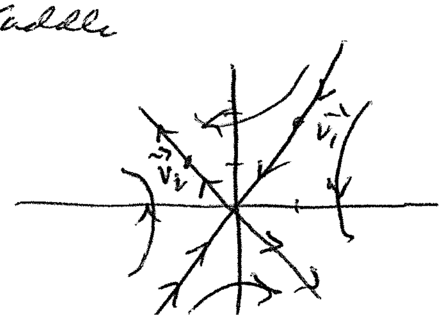
2 c) $\begin{bmatrix} \alpha & 1 \\ \alpha & \alpha \end{bmatrix}$ $Tr(A) = 2\alpha$
$\det(A) = \alpha^2 - \alpha$
$\Delta = 4\alpha^2 - 4(\alpha^2 - \alpha) = 4\alpha$

Sign analysis:

		$\alpha = 0$		$\alpha = 1$	
$Tr(A) = 2\alpha$	$-$		$+$		$+$
$\det(A) = \alpha(\alpha - 1)$	$+$		$-$		$+$
$\Delta = 4\alpha$	$-$		$+$		$+$
	~~Stable~~ Spiral sink	Uniform motion	Saddle	line of unstable fixed pts.	source

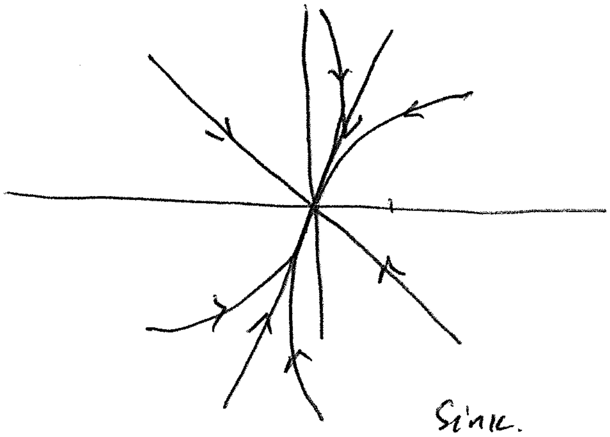
3) (a) $\lambda_1 = -1$ $\lambda_2 = 2$ ← Saddle

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



3(b) $\lambda_1 = -1$ $\lambda_2 = -2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

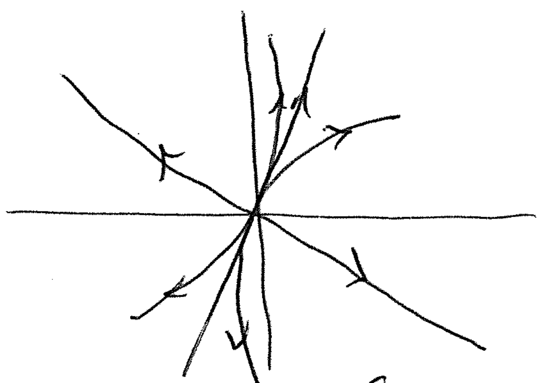


Sol'ns become tangent to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as $t \to \infty$.

Sink.

3(c) $\lambda_1 = 1$ $\lambda_2 = -2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$



Reverse the arrows from the sink.

Source.

4. (a) $\cancel{\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}}$ ~~Tr(A) = 4~~ ~~det(A) = 3+8 = 11~~ $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ $Tr(A) = 2$

$\det(A) = -3+8 = 5$

$\lambda^2 - 2\lambda + 5 = 0$

$(\lambda - 1)^2 = -4$

$\lambda = 1 \pm 2i$

$(3 - (1+2i))v_1 - 2v_2 = 0$

$(2-2i)v_1 - 2v_2 = 0$ $\begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \vec{v}$

$$e^{\lambda t}\vec{v} = e^{(1+2i)t}\begin{bmatrix}1\\1-i\end{bmatrix}$$

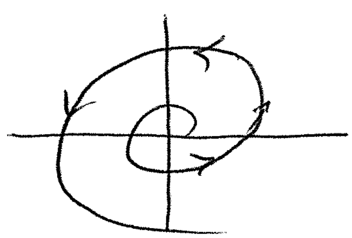
$$= e^{t}(\cos(2t)+i\sin(2t))\begin{bmatrix}1\\1-i\end{bmatrix}$$

$$= e^{t}\begin{bmatrix}\cos(2t) + i\sin(2t)\\ \cos(2t)+\sin(2t)+i(-\cos(2t)+\sin(2t))\end{bmatrix}$$

$$\vec{x}(t) = c_1 \text{ real}(e^{\lambda t}\vec{v}) + c_2 \text{ imag}(e^{\lambda t}\vec{v})$$

$$= e^{t}\left(c_1\begin{bmatrix}\cos(2t)\\ \cos(2t)+\sin(2t)\end{bmatrix} + c_2\begin{bmatrix}\sin(2t)\\ -\cos(2t)+\sin(2t)\end{bmatrix}\right)$$

Spiral source; for direction: $\begin{bmatrix}3 & -2\\4 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$



spiral source.

+ (b) $\begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}$ $\left.\begin{array}{l}Tr(A)=0\\ det(A) = -4+3=-1\end{array}\right\}$ saddle

$\lambda^2 - 1 = 0$

$\lambda = \pm 1 \Rightarrow$

$\lambda = 1$

$(2-1)v_1 - v_2 = 0$

$v_1 - v_2 = 0$

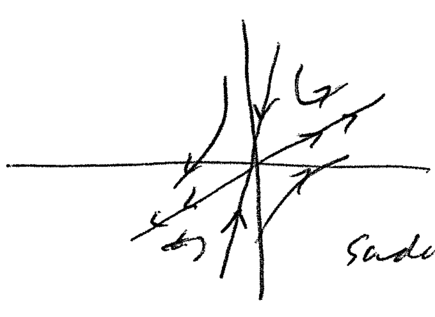
$\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$

$\lambda = -1$

$(2+1)v_1 - v_2 = 0$

$3v_1 - v_2 = 0$

$\vec{v} = \begin{bmatrix}1\\3\end{bmatrix}$



saddle

$$\vec{x} = c_1 e^{t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\3\end{bmatrix}$$

4(c) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ $Tr(A) = 0$

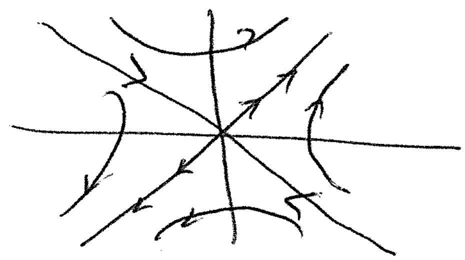
$\det(A) = -4$

$\lambda^2 - 4 = 0$

$\lambda = \pm 2$ (saddle)

$\lambda = 2$:

$-2v_1 + 2v_2 = 0$ $\quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



saddle.

$\lambda = -2$

$2v_1 + 2v_2 = 0$ $\quad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$