M244
Spr 2019

Quiz 11

Name: Icy

This is a take home quiz. Please write complete solutions (your own paper) to the following and turn in at the beginning of class on Monday.

1. Given the solution to $\mathbf{x}' = A\mathbf{x}$ for each matrix below (using eigenvalues and eigenvectors).

   (a) $A = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} -2 & -2 \\ 5 & 0 \end{bmatrix}$ (c) $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

2. For the first two systems in Question 1, provide a sketch of the phase plane (the (x_1, x_2) plane) and some solutions.

3. For the nonlinear system below, (i) find the equilibrium solutions, then for each equilibrium, (ii) find the best approximate matrix system, and (iii) Classify the equilibrium (Poincare).

$$\begin{aligned} x' &= x(1 - (1/2)x - (1/2)y) \\ y' &= y(-(1/4) + (1/2)x) \end{aligned}$$

   (Hint: There are three equilibria)

1(a) $A = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$ $Tr(A) = 1$

$det(A) = -2$ saddle

$\lambda^2 - \lambda + 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0 \quad \lambda = -1, 2$

$\lambda = -1$

$v_1 - v_2 = 0 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$

$-2v_1 - v_2 = 0 \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(b) $A = \begin{bmatrix} -2 & -2 \\ 5 & 0 \end{bmatrix}$ $Tr(A) = -2$

$det(A) = 10$

Spiral sink

$\lambda^2 + 2\lambda + 1 + 9 = 0$

$(\lambda + 1)^2 = -9$

$\lambda = -1 \pm 3i$

$\lambda = -1 + 3i$

$(-2 - (-1 + 3i))v_1 - 2v_2 = 0$

$(-1 - 3i)v_1 - 2v_2 = 0$

$\vec{v} = \begin{bmatrix} -2 \\ 1 + 3i \end{bmatrix}$

$$e^{\lambda t}\vec{v} = e^{-t}(\cos(3t) + i\sin(3t)) \begin{bmatrix} -2 \\ 1 + 3i \end{bmatrix} = e^{-t} \begin{bmatrix} -2\cos(3t) + i(-2\sin 3t) \\ \cos(3t) - 3\sin(3t) + i(3\cos(3t) + \sin(3t)) \end{bmatrix}$$

$$\vec{x}(t) = e^{-t}\left(c_1 \begin{bmatrix} -2\cos(3t) \\ \cos(3t) - 3\sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} -2\sin(3t) \\ 3\cos(3t) + \sin(3t) \end{bmatrix} \right)$$

c) $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ $Tr(A) = 6$

$Det(A) = 8 - -1 = 9$ $\quad \lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

degen source

$\lambda = 3, 3$

We use initial condition $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$:

$(2 - 3)x_0 + y_0 = w_1$

$-x_0 + (4 - 3)y_0 = w_2 \Rightarrow \vec{w} = \begin{bmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{bmatrix}$

$\Rightarrow$

$$\vec{x}(t) = e^{3t}\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{bmatrix} \right)$$

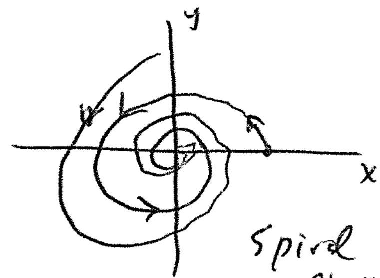
2) (a)

Saddle

(b) Note: $\begin{bmatrix} -2 & -2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$\uparrow$

CCW



Spiral sink

3. $x' = x\left(1 - \frac{1}{2}x - \frac{1}{2}y\right)$

$y' = y\left(-\frac{1}{4} + \frac{1}{2}x\right)$

i) Equil: Set $x' = y' = 0$, solve for x, y:

From Eqn 1, $x = 0$ or $2 - y = x$

If $x = 0$, in eqn 2, $y = 0$

If $x = 2 - y$, then $y\left(-\frac{1}{4} + \frac{1}{2}(2-y)\right) = 0$

so $y = 0$ (and $x = 2$)

or $y = \frac{3}{2}$ (and $x = 1/2$)

3 Equil: $(0,0)$, $(2,0)$, and $(1/2, 3/2)$.

(ii) Jacobian Matrix

$$\begin{bmatrix} 1 - x - \frac{1}{2}y & -\frac{1}{2}x \\ \frac{1}{2}y & -\frac{1}{4} + \frac{1}{2}x \end{bmatrix}$$

At $(0,0)$: $\begin{bmatrix} 1 & 0 \\ 0 & -1/4 \end{bmatrix}$ Saddle

At $(2,0)$: $\begin{bmatrix} -1 & -1 \\ 0 & 3/4 \end{bmatrix}$ Saddle

At $(1/2, 3/2)$: $\begin{bmatrix} -1/4 & -1/4 \\ 3/4 & 0 \end{bmatrix}$ Spiral sink

(see plot attached.)

How to draw the actual direction field for the last quiz problem using Maple. (I just wanted you to try to hand-sketch).

```
> f:=(x,y)->x*(1-(1/2)*x-(1/2)*y); g:=(x,y)->y*(-(1/4)+(1/2)*x);
```

$$f := (x, y) \to x\left(1 - \frac{1}{2}x - \frac{1}{2}y\right)$$

$$g := (x, y) \to y\left(-\frac{1}{4} + \frac{1}{2}x\right) \tag{1}$$

```
> MLV:=[diff(x(t),t)=f(x(t),y(t)), diff(y(t),t)=g(x(t),y(t))];
```

$$MLV := \left[\frac{d}{dt}x(t) = x(t)\left(1 - \frac{1}{2}x(t) - \frac{1}{2}y(t)\right), \frac{d}{dt}y(t) = y(t)\left(-\frac{1}{4} + \frac{1}{2}x(t)\right)\right] \tag{2}$$

```
> with(plots): with(DEtools):
  ivs:=[[x(0)=1,y(0)=0.5],[x(0)=1,y(0)=5/2]];
```

$$ivs := \left[[x(0) = 1, y(0) = 0.5], \left[x(0) = 1, y(0) = \frac{5}{2}\right]\right] \tag{3}$$

```
> DEplot(MLV,[x(t),y(t)],t=0..10,ivs,x=-0.5..3,y=-0.5..3);
```