Sections 3.8

Last time, we focused our attention on the spring-mass model that had no forcing. Today, we look at the special case of **periodic forcing.**

No Damping, Periodic Forcing

Rather than write the model as $mu'' + ku = F(t)$, it is common practice to write the model as $u'' + \omega_0^2 u = F(t)$. That way it is easy to read off the circular frequency of the homogenous part of the solution. Putting in the periodic forcing (we'll use cosine)

$$u'' + \omega_0^2 u = F_0 \cos(\omega t)$$

And, as long as the forcing function does not have the same frequency as the natural frequency, $\omega_0 \neq \omega$, then we can solve for the particular solution using complexification: $y_p = A\mathrm{e}^{i\omega t}$:

$$A\mathrm{e}^{i\omega t}(-\omega^2 + \omega_0^2) = F_0 \mathrm{e}^{i\omega t} \quad \Rightarrow \quad A = \frac{F_0}{\omega_0^2 - \omega^2}$$

so the overall solution is:

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{w_0^2 - w^2} \cos(wt)$$

It can be shown (Exercise 1) that, if $y(0) = y'(0) = 0$, then

$$C_1 = -\frac{F_0}{\omega_0^2 - \omega^2} \qquad C_2 = 0$$

In this case,

$$y(t) = \frac{F_0}{w_0^2 - w^2} \left(\cos(wt) - \cos(w_0 t)\right)$$

In the handout on the next page, we see what happens to this function. We see a series of graphs where $w_0 = 1$ and w is changing, from $w = 2$ to $w = 1.01$. The values were $w = 2, 1.5, 1.1, 1.01$

This shows the phenomena known as **beating**. Beating occurs (formally) when there is no damping, and when the frequency for the driving force is very close to (but not equal to) the natural frequency.

Using some trig identities, it is possible to rewrite $y(t)$ as the following (p. 213 of the text):

$$\frac{2F_0}{\omega_0^2 - \omega^2} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right) \sin\left(\frac{(\omega_0 + \omega)t}{2}\right)$$

As $\omega \to \omega_0$, the period of the first sine gets larger and larger. In fact, we see the large, slower oscillating function in the images below- They represent

$$\pm\frac{2F_0}{\omega_0^2 - \omega^2} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right)$$

From the images, we see that ONE SINGLE BEAT has CIRCULAR FREQUENCY that is half of the sine wave, or $|\omega_0 - \omega|$.

From Beating to Resonance

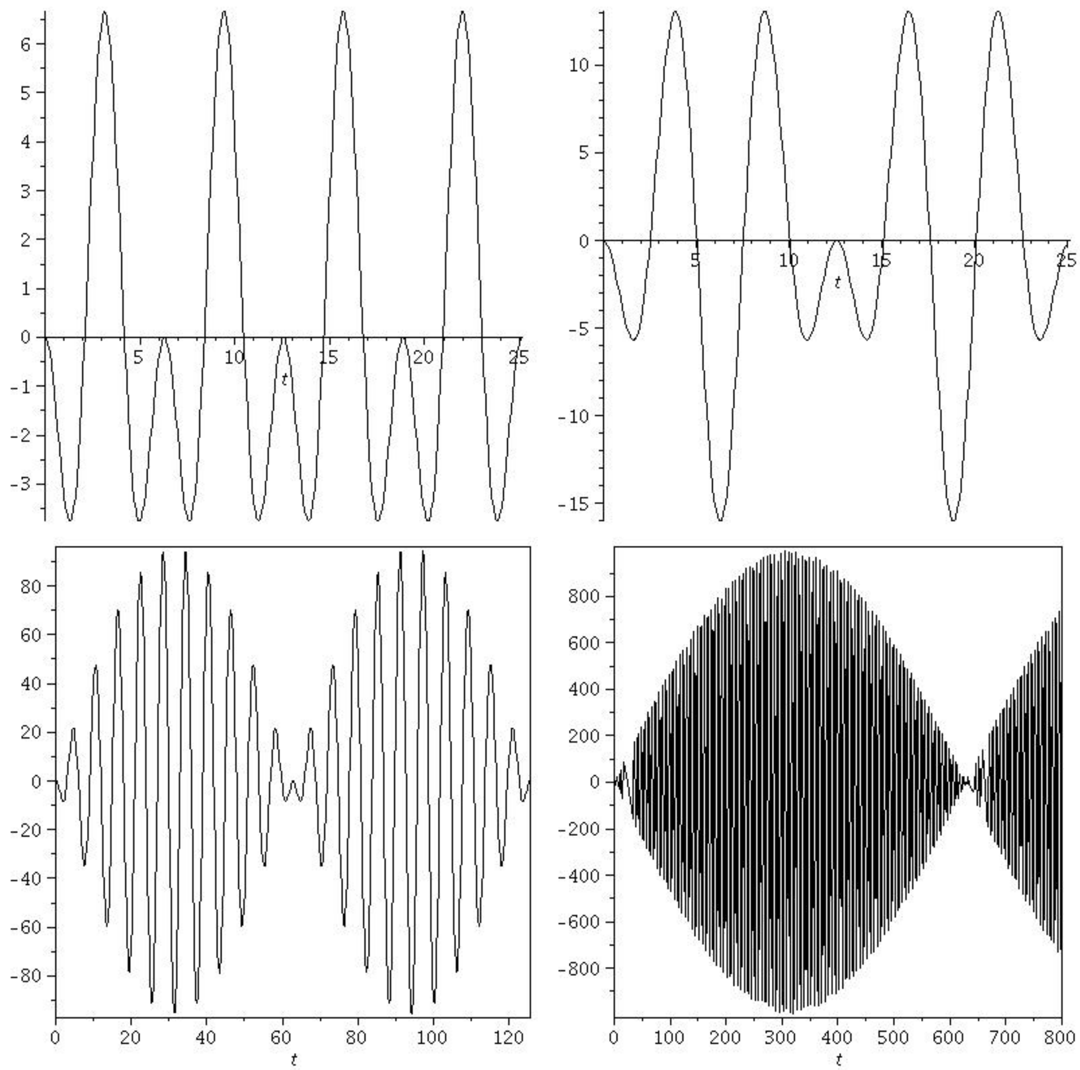


Figure 1: Figure showing the function $\frac{10}{1-\omega^2}(\cos(t) - \cos(\omega t))$. The graphs show the result of taking $\omega = 2$, $\omega = 1.5$, $\omega = 1.1$ and $\omega = 1.01$. As $w \to 1$, we see that the amplitude and period of the beats are getting larger and larger and larger!

As a particular example, consider the upper right figure. This comes from taking $\omega_0 = 1$ and $\omega = 1.5$. In that case, the circular frequency is $1/2$, which makes the period of one beat: $2\pi/(1/2) = 4\pi \approx 12.5$ (examine the figure to double check).

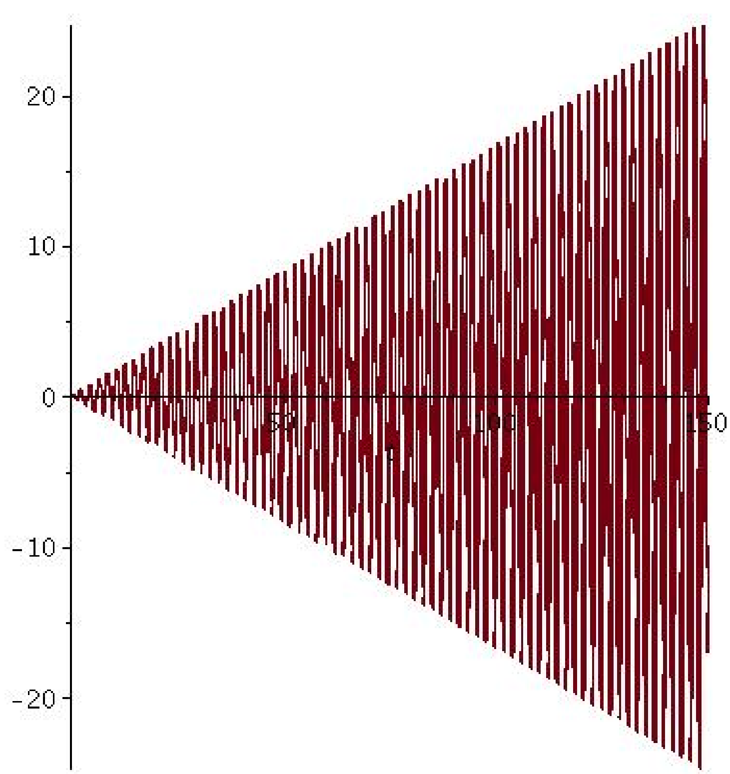
Resonance

What happens at $\omega = \omega_0$? Something known as **resonance**.

- Geometrically, we see that the period of the larger beat is infinitely long, with an infinitely large amplitude (draw graph).

- Algebraically (from Method of Undetermined Coefficients), we see that the new guess for the particular solution:
$$u'' + w_0^2 u = F_0 \cos(w_0 t)$$
is $y_p(t) = tA\mathrm{e}^{i\omega_0 t}$. We could substitute this in and solve, or we can take the limit of our solution.

- Using our previous solution, we take the limit using L'Hospital's rule:
$$\lim_{w \to w_0} \frac{F_0\left(\cos(wt) - \cos(w_0 t)\right)}{w_0^2 - w^2} = \lim_{\omega \to \omega_0} \frac{-F_0 t \sin(\omega t)}{-2\omega} = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$



Bringing in Damping: The Full Model

When we have the full damped system, then all solutions will tend to zero as $t \to \infty$. How do we know that? Given

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

We will never have purely periodic solutions for y_h, therefore, y_p will never have the multiplication by t (by undetermined coefficients). HOWEVER, we will still analyze a slightly changed version of the system:

$$u'' + pu' + qu = \cos(\omega t)$$

Now, we'll feel free to make $y_p = A\mathrm{e}^{i\omega t}$, and substitute getting:

$$A\mathrm{e}^{i\omega t}(-\omega^2 + i\omega p + q) = \mathrm{e}^{i\omega t}$$

And we see that:

$$A = \frac{1}{(q - \omega^2) + i\omega p}$$

Since the forcing function is a cosine function, from our handout, we had that the amplitude R and phase angle δ for y_p are given by:

$$R = \frac{1}{|(q - \omega^2) + iwp|} \qquad \delta = \tan^{-1}\left(\frac{\omega p}{q - \omega^2}\right)$$

Computing these, we can say that the forcing function then has the form

$$y_p = R\cos(\omega t - \delta)$$

Now, suppose that we can "tune" the value of ω, so that we'll fix the other model parameters p, q. In that case, R becomes a function of ω:

$$R = \frac{1}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}}$$

And we can ask: Is there a value of ω that **maximizes** the amplitude of the forced response, R? To find the max in this case, we'll differentiate R, then set the derivative to zero. Before we do that, notice the following:

$$R = \frac{1}{\sqrt{f(\omega)}} = (f(\omega))^{-1/2} \quad \Rightarrow \quad \frac{dR}{d\omega} = -\frac{1}{2}(f(\omega))^{-3/2} f'(\omega) = -\frac{1}{2}\,\frac{f'(\omega)}{(f(\omega))^{3/2}}$$

If we set $\frac{dR}{d\omega} = 0$, we just have $f'(\omega) = 0$- So that simplifies our computation by quite a bit.

$$f(\omega) = (q - \omega^2)^2 + p^2\omega^2 \quad \Rightarrow \quad \frac{df}{d\omega} = 2(q - \omega^2)(-2\omega) + p^2 \cdot 2\omega = 0$$

Solving for only the positive ω, we get $\omega = \sqrt{\frac{2q - p^2}{2}}$. It is at this maximizer that we'll extend the idea of **resonance** to cover... Here, the amplitude of the response can be VERY LARGE, even though damping is present.

Homework
Replaces Section 3.8

1. Solve the IVP $u'' + \omega_0^2 u = F_0 \cos(\omega t)$, $u(0) = 0$ and $u'(0) = 0$, if $\omega \neq \omega_0$.

2. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$.

   NOTE: We defined L to be the length of the spring stretched from its natural length to equilibrium.

3. Convert the following to $R\cos(\omega_0 t - \delta)$.

   (a) $\cos(9t) - \sin(9t)$

   (b) $2\cos(3t) + \sin(3t)$

   (c) $-2\pi\cos(\pi t) - \pi\sin(\pi t)$

   (d) $5\sin(t/2) - \cos(t/2)$

4. Suppose $u'' + 4u = \cos(2.8t)$. This function exhibits beating. (i) Give the frequence and amplitude of a beat, and (ii) Give (only) the particular part of the solution. HINT: Did you notice the relationship between the constant in front of the particular solution and the constant in front of the product of sines on page 1?

5. Same question as before, but with $u'' + 9u = \cos(3.1t)$.

6. Same question as before, but with $u'' + u = \cos(1.3\,t)$

7. Find the solution to $u'' + 9u = \cos(3t)$ with zero initial conditions.

8. Find the general solution using complex exponentials: $y'' + 3y' + 2y = \cos(t)$.

9. Consider $u'' + pu' + qu = \cos(\omega t)$. In the notes at the bottom of p. 4, we got that

$$\omega = \sqrt{\frac{2q - p^2}{2}}$$

   Thinking of p as damping, if the damping is very very small, then approximately what value of ω will result in a very large amplitude response?

10. Consider $u'' + u' + 2u = \cos(t)$. Find the amplitude and the phase angle of the particular part of the solution (or, for the forced response). You do NOT need to get the homogeneous part of the solution.

11. Consider $u'' + u' + 2u = \cos(\omega t)$. Find the value of ω that will maximize the amplitude of the response.

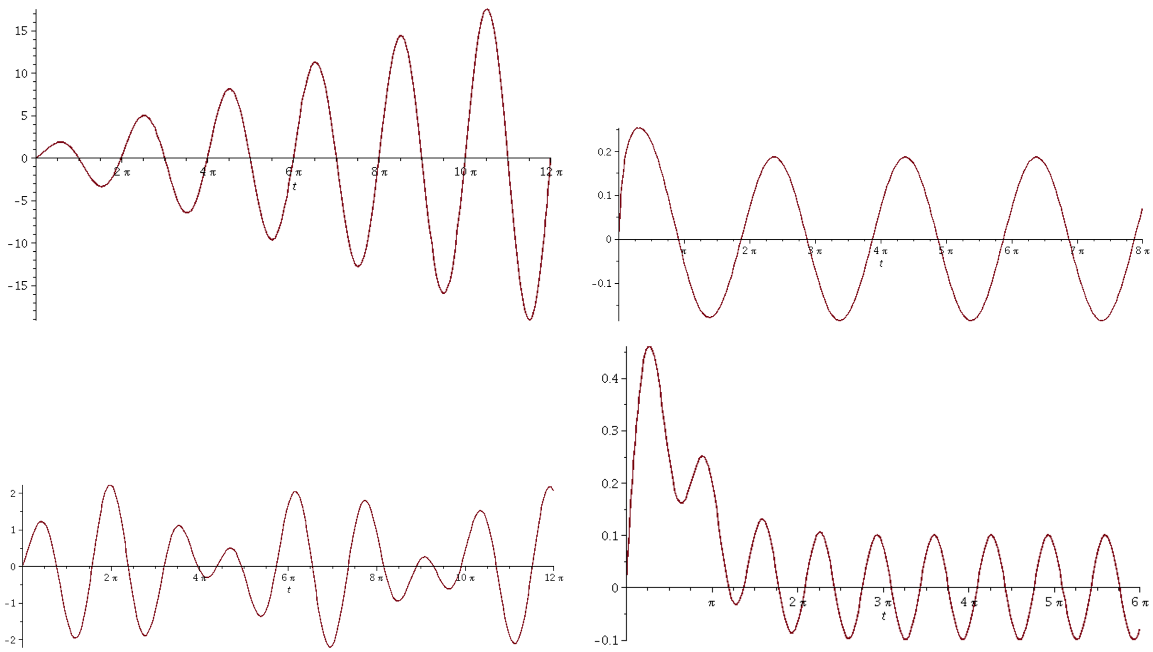
    NOTE: I don't want you to memorize the value of ω. Rather, find the amplitude R, then differentiate to find where the derivative is zero. Remember our shortcut (dealing with $f(\omega)$).

12. Pictured below are the graphs of several solutions to the differential equation:

$$y'' + py' + qy = \cos(\omega t)$$

Match the figure to the choice of parameters.

Choice	b	c	ω
(A)	5	3	1
(B)	0	2	1
(C)	0	1	1
(D)	2	1	3



13. Write the forced response to the ODE below as $R\cos(\omega t - \delta)$:

$$u'' + u' + 2u = \cos(3t)$$

14. Suppose we can tune the value of q rather than the value of ω in the differential equation (where $\omega = 3$):

$$u'' + u' + qu = \cos(3t)$$

Find the value of q that will maximize the amplitude of the forced response.