# Lecture Notes to substitute for 7.3-7.5

We want to solve the system:

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \Rightarrow \quad \mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}$$

SOLUTION: Use the ansatz $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$.

Then $\mathbf{x}' = \lambda \mathrm{e}^{\lambda t}\mathbf{v}$, so that the DE becomes:

$$A\mathrm{e}^{\lambda t}\mathbf{v} = \lambda \mathrm{e}^{\lambda t}\mathbf{v} \quad \Rightarrow A\mathbf{v} = \lambda\mathbf{v} \quad \text{or} \quad \begin{aligned} av_1 + bv_2 = \lambda v_1 \\ cv_2 + dv_2 = \lambda v_2 \end{aligned}$$

If the system above is true for that particular value of $\lambda$ and **non-zero** vector $\mathbf{v}$, then $\lambda$ is an **eigenvalue** of the matrix $A$ and $\mathbf{v}$ is an associated **eigenvector**. Note that while $\mathbf{v}$ is not allowed to be the zero vector, $\lambda$ could be zero.

## Computing Eigenvalues and Eigenvectors:

Consider the system we had:

$$\begin{aligned} av_1 + bv_2 &= \lambda v_1 \\ cv_1 + dv_2 &= \lambda v_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (a-\lambda)v_1 \quad &+bv_2 &= 0 \\ cv_1 \quad &+(d-\lambda)v_2 &= 0 \end{aligned}$$

That is the key system of equations. We saw last time $A\mathbf{x} = 0$ has exactly the zero solution iff $\det(A) \neq 0$. Therefore, for this system to have a **non-trivial solution** (which is a non-zero eigenvector), the **determinant must be zero**.

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

You might recognize those two quantities that are computed as the trace and determinant of $A$:

$$\mathrm{Tr}(A) = a + d \qquad \det(A) = ad - bc$$

**Theorem:** The eigenvalues for the $2 \times 2$ matrix $A$ are found by solving the **characteristic equation**:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

So, given $A$, compute the $\mathrm{Tr}(A)$, the $\det(A)$ and the discriminant,

$$\Delta = (\mathrm{Tr}(A))^2 - 4\det(A)$$

Then the eigenvalues are:

$$\lambda = \frac{Tr(A) \pm \sqrt{\Delta}}{2}$$

Just as in Chapter 3, the form of the solution will depend on whether $\Delta$ is positive (two real $\lambda$), negative (two complex $\lambda$) or zero (one real $\lambda$). Today, we will **focus on the distinct eigenvalues** case.

## Examples: Solve for eigenvalues and eigenvectors

Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find eigenvalues and eigenvectors for $A$.

SOLUTION: We could jump right to the characteristic equation, but for practice its good to write down what it is we actually want to solve (the unknowns below are $\lambda, v_1, v_2$):

$$\begin{array}{rl} 7v_1 + 2v_2 & = \lambda v_1 \\ -4v_1 + v_2 & = \lambda v_2 \end{array} \quad \Rightarrow \quad \begin{array}{rl} (7-\lambda)v_1 + 2v_2 & = 0 \\ -4v_2 + (1-\lambda)v_2 & = 0 \end{array}$$

For this to have a non-zero solution $v_1, v_2$, the determinant must be zero:

$$(7-\lambda)(1-\lambda) + 8 = 0 \quad \Rightarrow \quad \lambda^2 - 8\lambda + 15 = 0$$

(Note that the trace is 8, determinant is 15). This factors, so we can solve for $\lambda$:

$$(\lambda - 5)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda = 3, 5$$

Now, for each $\lambda$, go back to our system of equations for $v_1, v_2$ and solve. NOTE: By design, these equations should be multiples of each other!

For $\lambda = 3$:

$$\begin{array}{rl} (7-3)v_1 + 2v_2 & = 0 \\ -4v_2 + (1-3)v_2 & = 0 \end{array} \quad \Rightarrow \quad 4v_1 + 2v_2 = 0$$

There are an infinite number of solutions (and there always be). We want to choose “nice” values of $v_1, v_2$ that satisfies this relationship (or alternatively, lies on the line). An easy way of writing $v_1, v_2$ is to notice the following:

Given $ax + by = 0$, we can choose $x = b, y = -a$ to lie on the line.

Continuing, we'll take our vector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Now go through the same process to find the eigenvector for $\lambda = 5$:

$$\begin{array}{rl} (7-5)v_1 + 2v_2 & = 0 \\ -4v_1 + (1-5)v_2 & = 0 \end{array} \quad \Rightarrow \quad 2v_1 + 2v_2 = 0 \quad \Rightarrow \quad v_1 + v_2 = 0$$

Therefore, we take $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

An important note: If $\mathbf{v}$ is an eigenvector, then so is any scalar multiple of $\mathbf{v}$ (that is, the set of all eigenvectors forms a line). Therefore, when computing eigenvectors by hand, we typically re-scale them so that they are integers.

# Example: Solve a Linear System

Solve the linear system using eigenvectors:

$$\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$$

SOLUTION: We find the eigenvalues $\lambda_1, \lambda_2$, and the corresponding eigenvectors. Then the solution is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t} \mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t} \mathbf{v}_2$$

With that, we first compute the eigenvalues and eigenvectors. The determinant is 8, the trace is 6. The characteristic equation is:

$$\lambda^2 - 6\lambda + 8 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda - 4) = 0$$

Therefore, $\lambda = 2, 4$.

- For $\lambda = 2$:

$$\begin{array}{ll} (3-2)v_1 + v_2 & = 0 \\ v_1 + (3-2)v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{ll} v_1 + v_2 & = 0 \\ v_1 + v_2 & = 0 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

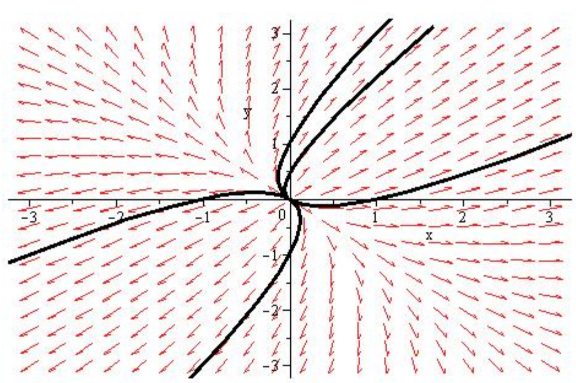
- For $\lambda = 4$,

$$\begin{array}{ll} (3-4)v_1 + v_2 & = 0 \\ v_1 + (3-4)v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{ll} -v_1 + v_2 & = 0 \\ v_1 - v_2 & = 0 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \mathrm{e}^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Visualization of the solutions. Can you locate the lines created by the two eigenvectors?



## Example: Solve the linear system.

The technique is the same as the last example. Write it down, try it out, then come back to this page to see if we have the same answer.

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} \qquad \text{Tr}(A) = 1 \quad \det(A) = -2 \quad \Delta = 9$$

The characteristic equation is $\lambda^2 - \lambda - 2 = 0$, or $(\lambda + 1)(\lambda - 2) = 0$.

The eigenvalues are $\lambda = -1, 2$. The corresponding eigenvectors are found by solving the system above. For $\lambda = -1$:

$$\begin{aligned} (3+1)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2+1)v_2 &= 0 \end{aligned} \qquad 2v_1 - v_2 = 0 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = 2$:

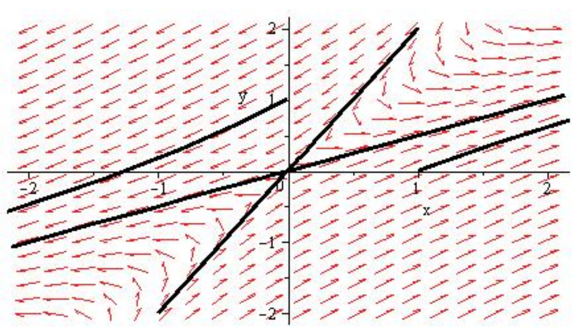
$$\begin{aligned} (3-2)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2-2)v_2 &= 0 \end{aligned} \qquad v_1 - 2v_2 = 0 \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution to the system of differential equations is then:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

To sketch the graph of the solution, first locate the two lines created by the eigenvectors. You might notice that if the corresponding eigenvalue is negative, the solution then moves along the line to the origin (otherwise, the solution moves along the line to infinity).

The origin is the only equilibrium solution, and in this case (with mixed signs of the eigenvalues), it is called a **saddle point**.




## Some definitions: Classifying the origin

Given $\mathbf{x}' = A\mathbf{x}$, the origin is an equilibrium solution.

- If the eigenvalues are both positive, the origin is a **source**.
- If the eigenvalues are both negative, the origin is a **sink**.
- If the eigenvalues are mixed in sign, the origin is a **saddle**.